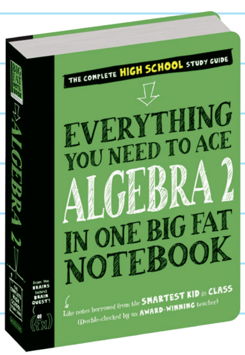
EVERYTHING YOU NEED TO ACE ALGEBRA 2 IN ONE BIG FAT NOTEBOOK

20 EXTRA CHAPTERS TO SUPPLEMENT THE BOOK!



WORKMAN PUBLISHING
NEW YORK

Workman Kids
Workman Publishing
Hachette Book Group, Inc.
1290 Avenue of the Americas
New York, NY 10104
workman.com

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This material supplements *Everything You Need to Ace Algebra 2 In One Big Fat Notebook* (ISBN 978-1-5235-2592-8) published in September 2025.

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EVERYTHING YOU NEED TO ACE

ALGEBRA 2

BONUS CHAPTERS!

Algebra 2 is a huge topic. *Everything You Need to Ace Algebra 2 In One Big Fat Notebook* covers all the essential stuff, but there's always a chance you might come across a skill that didn't make it in the book. Don't worry! We have you covered there, too. These are some extra chapters that explain everything else you might need to know.

EQUATIONS and INEQUALITIES

MATRICES

SEQUENCES and SERIES

PROBABILITY

STATISTICS

TRIGONOMETRY

EXPONENTIAL FUNCTIONS IN THE REAL WORLD

Exponential functions are used to model certain real-life situations. These include **EXPONENTIAL GROWTH**, such as population growth and dollars earned from **COMPOUND INTEREST**, as well as **EXPONENTIAL DECAY**, such as the breakdown of radioactive materials.

COMPOUND INTEREST
interest earned on both the initial amount of money deposited as well as the accumulated interest from previous periods

To calculate compound interest, we use one of the following formulas.

For interest compounded n times per year:

$$A = P\left[1 + \frac{r}{n}\right]^{nt}$$

For interest compounded continuously:

$$A = Pe^{rt}$$

In these formulas:

A = total amount of money in dollars

P = the amount of the initial investment

r = the rate of interest per year expressed as a decimal or fraction

n = the number of times the interest is compounded each year

t = the number of years of the investment

e = Euler's number (approximately 2.71828)

Let's look at some examples of real-world situations that require the use of exponential functions.

EXAMPLE: Katelyn receives $400,000 from an investor to purchase supplies for her business. She decides that she will first deposit that money in an account that offers 8% interest compounded quarterly and leave it there for 1 year. To the nearest cent, how much money will Katelyn have in this account after 1 year?

To solve this problem, use the formula for interest compounded n times per year:

$$A = P\left[1 + \frac{r}{n}\right]^{nt}$$

Let P = 400,000 (initial investment in dollars).
Let r = 0.08 (8% rate).
Let n = 4 (the interest is compounded quarterly, meaning 4 times per year).
Let t = 1 (time in years).
Let A = total account balance in dollars (unknown).

$$A = P\left[1 + \frac{r}{n}\right]^{nt}$$ *Substitute the given values.*

$$A = 400{,}000\left[1 + \frac{0.08}{4}\right]^{4(1)}$$

$$A = 400{,}000\left[1 + \frac{0.08}{4}\right]^{4}$$ *Simplify.*

$$A = 400{,}000[1 + 0.02]^4$$

$$A = 400{,}000\,[1.02]^4$$

$$A \approx 432{,}972.86$$ *Use a scientific or graphing calculator to evaluate. Round to the nearest cent.*

So, Katelyn will have $432,972.86 in her account after 1 year.

EXAMPLE: Howard wants to invest $50,000 in an account that offers 4.8% interest compounded monthly. His plan is to double his money to $100,000 to use toward the purchase of a new home. How many years will it take for Howard's account to reach at least $100,000?

To solve this problem, use the formula for interest compounded n times per year:

$$A = P\left[1 + \frac{r}{n}\right]^{nt}$$

Let P = 50,000 (initial investment in dollars).
Let r = 0.048 (4.8% rate).
Let n = 12 (the interest is compounded monthly, meaning 12 times per year).
Let t = time in years (unknown).
Let A = 100,000 (total account balance in dollars).

$A = P\left[1 + \frac{r}{n}\right]^{nt}$ Substitute the given values.

$100{,}000 = 50{,}000\left[1 + \frac{0.048}{12}\right]^{12t}$

$100{,}000 = 50{,}000\,[1 + 0.004]^{12t}$ Simplify.

$100{,}000 = 50{,}000\,[1.004]^{12t}$

$\frac{100{,}000}{50{,}000} = \frac{50{,}000\,[1.004]^{12t}}{50{,}000}$

$2 = [1.004]^{12t}$

$\log 2 = \log\,[1.004]^{12t}$ Take the common logarithm of each side of the equation.

$\log 2 = 12t \log 1.004$ Power Property of logarithms

$\frac{\log 2}{12 \log 1.004} = \frac{12t \log 1.004}{12 \log 1.004}$

$\frac{\log 2}{12 \log 1.004} = t$

$t \approx 14.4694$ Use a scientific or graphing calculator to evaluate.

Since it takes *more than* 14 years for Howard's account to reach \$100,000, Howard's account will reach *at least* \$100,000 after 15 years.

So, the answer is 15 years.

EXAMPLE: Stephanie deposits \$18,000 into an online savings account that pays 5% annual interest compounded continuously. If she makes no withdrawals, what is the balance to the nearest cent after 1 year?

To solve this problem, we can use the formula for interest compounded continuously:

$A = Pe^{rt}$

Let P = 18,000 (initial deposit in dollars).
Let r = 0.05 (5% rate).
Let t = 1 (time in years).
Let A = total account balance in dollars (unknown).

$A = Pe^{rt}$

$A = 18{,}000e^{0.05(1)}$ *Substitute the given values.*

$A \approx 18{,}922.88$ *Use a scientific or graphing calculator to evaluate. The button for e^x will be on the keypad. Round to the nearest cent.*

So, if Stephanie makes no withdrawals, she will have a balance of $18,922.88 in her account after 1 year.

EXAMPLE: The population of a city is modeled by the exponential equation $P = 11 \cdot 1.05^t$, where P is the population in millions after t years, with $t = 0$ corresponding to the year 2010. Approximately how many people will live in this city in the year 2040? Round the population to the nearest ten thousand people.

To solve this problem, substitute the value of t into the given exponential equation.

$t = 2040 - 2010 = 30$ ← *Value of t.*

$P = 11 \cdot 1.05^t$ ← *the given exponential equation*

$P = 11 \cdot 1.05^{30}$ *Substitute $t = 30$.*

$P \approx 47.54136613$ *Use a scientific or graphing calculator to evaluate.*

So, this city will have a population of approximately 47.54 million people (or, equivalently, 47,540,000 people) in the year 2040.

EXAMPLE: The Gerald family purchases a car for $38,000. Various reports state that the car will depreciate at a rate of 12% per year. Write a function that models the value V of the car after t years. Then find the approximate value of the car to the nearest cent in 4 years.

To write a function that models the value of this car, write an exponential depreciation equation:

$V = P \cdot (1 - r)^t$

Let $P = 38{,}000$ (initial value of the car in dollars).
Let $r = 0.12$ (12% rate).
Let $t = 4$ (time in years).
Let V = value of the car (unknown).

Function: $V = 38{,}000 \cdot (1 - 0.12)^t = 38{,}000 \cdot 0.88^t$

Note: Depreciation is a *percent decrease*. When we decrease by a rate of 12%, it's the same as taking a rate of (100 − 12)% = 88%.

Value in 4 years:

$V = 38{,}000 \cdot 0.88^4$ Substitute $t = 4$.

$V \approx 22{,}788.42$ Use a scientific or graphing calculator to evaluate. Round to the nearest cent.

So, a function that models the value of the car is $V = 38{,}000 \cdot 0.88^t$, and the approximate value of the car to the nearest cent in 4 years is $22,788.42.

EXAMPLE: A radiologist injects 16 milligrams of iodine-131 (I-131) into a patient's arm to conduct a contrast CT scan. The half-life of I-131 is approximately 8 days. To the nearest day, how long will it take until only 10 milligrams of I-131 remain in the patient's body?

Let's begin with a general exponential decay equation.

$y = xr^t$

x = milligrams of I-131 initially injected into the patient

t = time in days

y = milligrams of I-131 that are left in the patient's body

r = percentage of I-131 leaving the patient's body per day

Let $x = 16$ (milligrams of I-131 initially injected into patient).

So, the equation is $y = 16r^t$.

To use the equation, we must find the value of the rate, r. We can use the fact that the half-life of I-131 is approximately 8 days, which tells us that after 8 days ($t = 8$), only half of the I-131 ($y = 8$) remains.

$8 = 16r^8$ Substitute the values into the equation.

$$\frac{8}{16} = \frac{16r^8}{16}$$

$$\frac{1}{2} = r^8$$

Notice that this means half the substance remains after 8 days.

$$\sqrt[8]{\frac{1}{2}} = \sqrt[8]{r^8}$$ Take the positive eighth root of each side of the equation.

$$\sqrt[8]{\frac{1}{2}} = r$$

$r \approx 0.917$

We can now replace r by 0.917 in the equation: $y = 16 \cdot 0.917^t$

So, let's answer the original question.

Let t = time in days it takes for I-131 to leave the body (unknown).

Let y = 10 (milligrams of I-131 that are left in the patient's body).

$$10 = 16 \cdot 0.917^t$$

$$\frac{5}{8} = 0.917^t$$

$$\log\frac{5}{8} = \log 0.917^t$$ *Take the log of each side of the equation.*

$$\log\frac{5}{8} = t\log 0.917$$ *Apply the Power Property of logarithms.*

$$\frac{\log\frac{5}{8}}{\log 0.917} = t$$

$$t \approx 5.4243$$

So, to the nearest day, in 5 days there will be 10 milligrams of iodine-131 left in the patient's body.

CHECK YOUR KNOWLEDGE

Solve each of the following real-life situations using an exponential function.

1. What is the growth rate as a percentage for the exponential equation $y = 500 \cdot 0.064^t$?

2. The temperature of sweet potato pie is modeled by the equation $T = 72 \cdot 0.84^m + 53$, where T is the temperature m minutes after the sweet potato pie has been removed from the oven and placed on a cooling rack. To the nearest minute, how long will it take for the sweet potato pie to reach a temperature of 67.99 degrees?

3. A technology company deposits $1,000,000 in profits from an app for 1 year into a savings account that has 7% interest compounded monthly. To the nearest cent, how much money will this company have in this account after 1 year?

4. Ronald deposits $36,000 into a money market account that pays 8.2% annual interest compounded quarterly. If Ronald makes no withdrawals, what will be the balance in the account to the nearest cent after 2 years?

5. Harper needs to save $20,000 to purchase a car. She has $14,000 in savings, and she deposits that money into a savings account that has an 8.5% interest rate that is compounded continuously. After how many years will Harper's savings account have at least $20,000?

6. The use of pesticides is causing the population of a rare insect on a tropical island to decrease at a rate of 7% each year. If there are approximately 15,800 of these insects now, then according to this population model, what will be the approximate population of these insects in 10 years if the use of these pesticides continues?

7. An online health and nutrition company purchases 8,000 boxes of organic herbal tea. If in each month the company sells approximately 11% of the tea remaining from the previous month, approximately how many months will it take for the company to have less than 100 boxes in supply?

8. The radioactive isotope indium-111 has a half-life of approximately 3 days. If we have 132 grams of the chemical in a container, in how many days, to the nearest day, will we have only 9 grams?

ANSWERS

CHECK YOUR ANSWERS

1. 6.4%
2. 9 minutes
3. $1,072,290.08
4. $42,345.43
5. 5 years
6. 7,646 insects
7. 38 months
8. 12 days

SOLVING POLYNOMIAL EQUATIONS AND INEQUALITIES

POLYNOMIAL EQUATIONS

If we can factor a polynomial into a product of linear factors, then we can obtain all the **ZEROS** of the polynomial using the **FACTOR THEOREM**.

Let's look at an example.

$f(x) = 4(x + 2)^3(x - 8)^2(x + 6)$

Since the linear factors of the polynomial f are $x + 2$, $x - 8$, and $x + 6$, by the Factor Theorem, the zeros of f are $x = -2$, $x = 8$, and $x = -6$.

The **MULTIPLICITY** of a zero is the number of times the factor $x - a$ appears when the polynomial is fully factored.

In the polynomial $f(x) = 4(x + 2)^3(x - 8)^2(x + 6)$:

$x = -2$ is a zero of multiplicity 3.

$x = 8$ is a zero of multiplicity 2.

$x = -6$ is a zero of multiplicity 1.

Think: The exponent tells us the multiplicity.

The **FUNDAMENTAL THEOREM OF ALGEBRA** states that the number of zeros of a polynomial is equal to its degree.

NOTE: The zeros can be complex and we need to include the multiplicity when counting the number of zeros.

Together the Factor Theorem and the Fundamental Theorem of Algebra provide a nice way to solve many polynomial equations.

EXAMPLE: Find the solutions of the polynomial equation $-4x^2 - 35x = -(x^3 + 150)$.

Step 1: Write the polynomial equation in standard form.

STANDARD FORM:
- The right-hand side of the equation is equal to zero.
- Exponents are in descending order.

$-4x^2 - 35x = -(x^3 + 150)$

$-4x^2 - 35x = -x^3 - 150$

$x^3 - 4x^2 - 35x + 150 = 0$ ← *third-degree polynomial equation in standard form*

Step 2: Factor completely. Since this is a third-degree polynomial with integer coefficients, we will use the Rational Zeros Theorem and synthetic division.

By the Fundamental Theorem of Algebra, this equation has three zeros.

Start by using the Rational Zeros Theorem to find all possible rational zeros.

$P(x) = 1x^3 - 4x^2 - 35x + 150 = 0$

Constant term: 150
Leading coefficient: 1

Factors of 150: ±1, ±2, ±3, ±5, ±6, ±10, ±15, ±25, ±30, ±50, ±75, ±150

Factors of 1: ±1

All possible rational zeros: ±1, ±2, ±3, ±5, ±6, ±10, ±15, ±25, ±30, ±50, ±75, ±150

Remember: If a polynomial function has rational zeros, they will have the form $\frac{p}{q}$.

$$\frac{p}{q} = \frac{\text{factors of the constant term}}{\text{factors of the leading coefficient}}$$

Next, test the possible rational zeros by using them as divisors in synthetic division. There are so many possible rational zeros that we will perform synthetic division only until we find one divisor that results in no remainder.

Coefficients of $x^3 - 4x^2 - 35x + 150$: 1, −4, −35, and 150.

Note: For this example, we could also use a graphing calculator to find the zeros.

SYNTHETIC DIVISION TABLE

$\frac{p}{q}$	1	−4	−35	150
1	1	−3	−38	112
2	1	−2	−39	72
3	1	−1	−38	36
5	1	1	−30	0

No remainder!

So, $x - 5$ is a factor. By the Factor Theorem, 5 is a zero of the polynomial.

Now use the last row of the table, with 5 as the rational zero, to write the quotient polynomial:

$x^2 + x - 30$

Therefore, $x^3 - 4x^2 - 35x + 150 = (x - 5)(x^2 + x - 30)$.

We found one of the zeros: 5.

There are two more zeros. We can find these zeros by factoring the quadratic polynomial $x^2 + x - 30$.

$x^2 + x - 30 = (x - 5)(x + 6)$

Step 3: Apply the Zero-Product Property to find the zeros of the quadratic polynomial.

$x - 5 = 0$	$x + 6 = 0$
$x = 5$	$x = -6$

Therefore, $x^3 - 4x^2 - 35x + 150 = (x - 5)(x - 5)(x + 6)$.

Remember: Since $x - 5$ came up as a factor twice, we say that 5 is a double zero (or double root) or a zero of multiplicity 2.

So, the solutions of the polynomial equation are $x = 5$ and $x = -6$.

EXAMPLE: Find the solutions of the polynomial equation $x^4 - 5x^2 - 36 = 0$.

Step 1: Since the polynomial equation is already in standard form, factor completely using the Rational Zeros Theorem and synthetic division.

$P(x) = 1x^4 - 5x^2 - 36$

Constant term: −36

Leading coefficient: 1

Factors of −36: ±1, ±2, ±3, ±4, ±6, ±9, ±12, ±18, ±36

Factors of 1: ±1

All possible rational zeros: ±1, ±2, ±3, ±4, ±6, ±9, ±12, ±18, ±36

Next, test the possible rational zeros by using them as divisors in synthetic division.

Notice the polynomial does not have an x^3 term or an x term. Therefore, before dividing add placeholders.

Coefficients of $x^4 - 5x^2 - 36$: 1, 0, −5, 0, and −36

SYNTHETIC DIVISION TABLE

$\frac{p}{q}$	1	0	−5	0	−36
−1	1	−1	−4	4	−40
1	1	1	−4	−4	−40
−2	1	−2	−1	2	−40
2	1	2	−1	−2	−40
−3	1	−3	4	−12	0

So, $x + 3$ is a factor. By the Factor Theorem, −3 is a zero of the polynomial.

Now use the last row of the table, with −3 as the rational zero, to write the quotient polynomial:

$x^3 - 3x^2 + 4x - 12$

Therefore, $x^4 - 5x^2 - 36 = (x + 3)(x^3 - 3x^2 + 4x - 12)$.

Keep factoring.

Step 2: Third-degree polynomial with integer coefficients:

$1x^3 - 3x^2 + 4x - 12$

Constant term: -12
Leading coefficient: 1

Factors of -12: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

Factors of 1: ± 1

All possible rational zeros in reduced form: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

Now use synthetic division to test the possible rational zeros again.

SYNTHETIC DIVISION TABLE

$\frac{p}{q}$	1	-3	4	-12
3	1	0	4	0

No remainder!

In the previous table, we found that 1, −1, 2, and −2 were not zeros, so we don't need to try them again.

So, $x - 3$ is a factor. By the Factor Theorem, 3 is a zero of the polynomial.

Use the last row of the table, with 3 as the rational zero, to write the quotient polynomial:

$x^2 + 4$

Therefore, $x^4 - 5x^2 - 36 = (x + 3)(x - 3)(x^2 + 4)$.

Step 3: Apply the Square Root Property to find the zeros of the quadratic polynomial $x^2 + 4 = 0$.

$x^2 + 4 = 0$

$x^2 = -4$

$x = \pm\sqrt{-4} = \pm 2i$

Note: The last two zeros are complex.

So, the solutions of the polynomial equation are $x = 3$, $x = -3$, $x = 2i$, and $x = -2i$.

POLYNOMIAL INEQUALITIES

A **POLYNOMIAL INEQUALITY** is a mathematical statement that contains polynomial expressions on both sides of an inequality symbol.

Examples of Polynomial Inequalities:

$x^2 - 5 > -x + 15$ $2x^3 + 4x < 2x^2$ $3x^3 + 3x^2 < 6x$

To solve a polynomial inequality, first solve the corresponding equation. These solutions are called the **CRITICAL VALUES** of the inequality.

EXAMPLE: Solve $x^2 - 3 > -x + 9$.

Step 1: Replace the inequality symbol with an equal sign.

$x^2 - 3 = -x + 9$

Step 2: Put the polynomial equation in standard form.

$x^2 + x - 3 = -x + x + 9$

$x^2 + x - 3 - 9 = 9 - 9$

$x^2 + x - 12 = 0$

Step 3: Solve the polynomial equation.

$x^2 + x - 12 = 0$

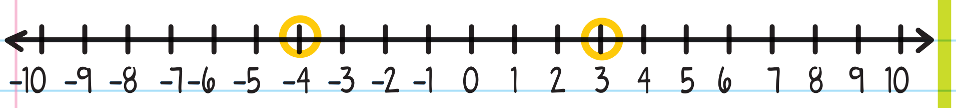
$(x + 4)(x - 3) = 0$

Step 4: Use the Factor Theorem to find the zeros.

$x = -4$ and $x = 3$

So, −4 and 3 are the critical values of the polynomial inequality.

Step 5: Plot each critical value on a number line. Since the inequality symbol in the original polynomial inequality is *greater than*, we use open circles at −4 and 3.

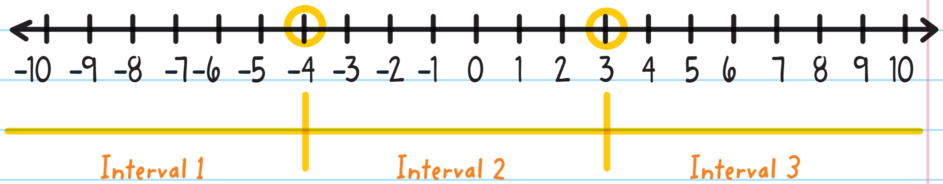


The critical values partition the number line into three intervals.

Interval 1: $x < -4$

Interval 2: $-4 < x < 3$

Interval 3: $x > 3$



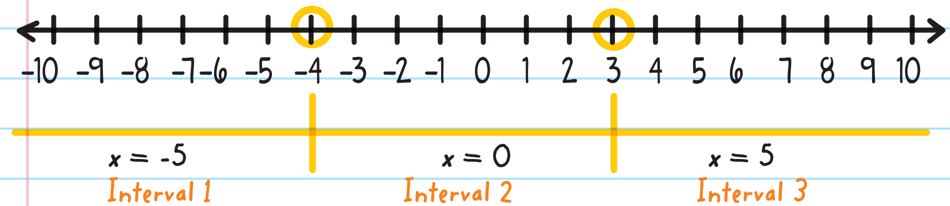
Step 6: Test a value from each interval to see if it satisfies the inequality.

Let's test –5 (for interval 1), 0 (for interval 2), and 5 (for interval 3).

$x^2 - 3 > -x + 9$	$x^2 - 3 > -x + 9$	$x^2 - 3 > -x + 9$
$(-5)^2 - 3 > -(-5) + 9$	$0^2 - 3 > -0 + 9$	$5^2 - 3 > -5 + 9$
$25 - 3 > 5 + 9$	$-3 > 9$ ✗ False	$25 - 3 > -5 + 9$
$22 > 14$ ✓ True		$22 > 4$ ✓ True

Think: The true statements tell us that all values in that interval ARE solutions to the polynomial inequality. The false statements tell us that all values in that interval are NOT solutions to the inequality.

Step 7: Shade the portions of the number line that satisfy the polynomial inequality. Then write the answer shown on the graph in interval notation.



Interval notation: $(-\infty, -4) \cup (3, \infty)$

So, the solution set of the polynomial inequality $x^2 - 3 > -x + 9$ is $(-\infty, -4) \cup (3, \infty)$.

CHECK YOUR KNOWLEDGE

For questions 1 through 5, find all solutions of the following polynomial equations.

1. $-3x^2 - 44x = -2x^3 + 60$

2. $x^3 + 2x = 3x^2 + 6$

3. $2x^3 + 2x = 8 + 3x^2$

4. $x^4 - 4x^3 - 7x^2 + 22x = -24$

5. $x^4 + 36 = 13x^2$

For questions 6 through 8, solve the following polynomial inequalities.

6. $x^3 + 6 \geq 5x + 2x^2$

7. $x^3 - 9x^2 > -8x$

8. $2x^4 - 3x^3 < 9x^2$

ANSWERS

CHECK YOUR ANSWERS

1. $x = -2, x = 6, x = -\frac{5}{2}$

2. $x = 3, x = \sqrt{2}\,i, x = -\sqrt{2}\,i$

3. $x = 2, x = -\frac{1}{4} + i\frac{\sqrt{31}}{4}, x = -\frac{1}{4} - i\frac{\sqrt{31}}{4}$

4. $x = -1, x = -2, x = 3, x = 4$

5. $x = 2, x = -2, x = 3, x = -3$

6. $[-2, 1] \cup [3, \infty)$

7. $(0, 1) \cup (8, \infty)$

8. $\left(-\frac{3}{2}, 0\right) \cup (0, 3)$

SOLVING RATIONAL EQUATIONS AND INEQUALITIES

An equation containing only rational expressions is called a **RATIONAL EQUATION**. A rational equation can have one solution, more than one solution, or no solution at all.

NOTE: Solving rational equations can lead to **EXTRANEOUS SOLUTIONS**. Therefore, it is important to do a careful CHECK at the end.

EXAMPLE: Solve $\frac{1}{5} + \frac{x}{x-2} = \frac{2}{x-2}$.

Step 1: Move all terms to the left side of the equation. Use the Properties of Equality.

$$\frac{1}{5} + \frac{x}{x-2} - \frac{2}{x-2} = 0$$

Step 2: Rewrite the terms using the least common denominator (LCD).

The LCD for this equation is $5(x - 2)$.

$$\frac{1(x-2)}{5(x-2)} + \frac{x(5)}{(x-2)(5)} - \frac{2(5)}{(x-2)(5)} = 0$$

$$\frac{x-2}{5(x-2)} + \frac{5x}{5(x-2)} - \frac{10}{5(x-2)} = 0$$

Step 3: Add.

$$\frac{6x-12}{5(x-2)} = 0$$

Step 4: Set the numerator equal to zero and solve for x.

$$6x - 12 = 0$$

$$x = 2$$

Remember: A fraction is zero whenever the numerator is zero and the denominator is not zero.

Step 5: Check to see if the solution is extraneous. Substitute the x-value in the denominator to see if it makes the denominator 0.

$5(x - 2)$
$= 5(2 - 2)$
$= 5 \cdot 0$
$= 0$

Therefore, 2 is an extraneous solution.

So, the rational equation $\frac{1}{5} + \frac{x}{x-2} = \frac{2}{x-2}$ has no solution.

EXAMPLE: Solve $\frac{x-1}{x+2} + \frac{x}{x+1} = \frac{6x+5}{x^2+3x+2}$.

Step 1: Move all terms to the left side of the equation, and factor each denominator.

$$\frac{x-1}{x+2} + \frac{x}{x+1} - \frac{6x+5}{x^2+3x+2} = 0$$

$$\frac{x-1}{x+2} + \frac{x}{x+1} - \frac{6x+5}{(x+2)(x+1)} = 0$$

Step 2: Find the LCD. Then rewrite each term using the LCD.

The LCD for this equation is $(x + 2)(x + 1)$.

$$\frac{(x-1)(x+1)}{(x+2)(x+1)} + \frac{x(x+2)}{(x+1)(x+2)} - \frac{6x+5}{(x+2)(x+1)} = 0$$

$$\frac{x^2-1}{(x+1)(x+2)} + \frac{x^2+2x}{(x+1)(x+2)} - \frac{6x+5}{(x+1)(x+2)} = 0$$

Step 3: Simplify.

$$\frac{2x^2-4x-6}{(x+1)(x+2)} = 0$$

Step 4: Set the numerator equal to zero and solve for x.

$2x^2 - 4x - 6 = 0$
$2(x^2 - 2x - 3) = 0$
$2(x - 3)(x + 1) = 0$

$x - 3 = 0$	$x + 1 = 0$
$x = 3$	$x = -1$

Step 5: Check to see if either solution is extraneous.

$x = 3$: $(x + 1)(x + 2) = (3 + 1)(3 + 2) = 4 \cdot 5 = 20 \neq 0$

$x = -1$: $(x + 1)(x + 2) = (-1 + 1)(-1 + 2) = 0 \cdot 1 = 0$

So, $x = 3$ is NOT extraneous, whereas $x = -1$ IS extraneous.

Therefore, $x = 3$ is the only solution to the rational equation.

So, the solution of the rational equation $\frac{x-1}{x+2} + \frac{x}{x+1} = \frac{6x+5}{x^2+3x+2}$ is $x = 3$.

An inequality containing only rational expressions is called a **RATIONAL INEQUALITY**.

We can solve a rational inequality just like we solve a rational equation. However, determining the solution set will require a little more work.

Let's look at an example.

EXAMPLE: Solve $\frac{3x+2}{x-4} > 2$.

Step 1: Rewrite the inequality as an equation.

$$\frac{3x+2}{x-4} = 2$$

$$\frac{3x+2}{x-4} - 2 = 0$$

Step 2: Rewrite each term using the LCD.

The LCD for this equation is $x - 4$.

$$\frac{3x+2}{x-4} - \frac{2}{1} \cdot \frac{x-4}{x-4} = 0$$

$$\frac{3x+2}{x-4} - \frac{2x-8}{x-4} = 0$$

$$\frac{x+10}{x-4} = 0$$

the zeros of the numerator and denominator

Step 3: Find the **critical values.**

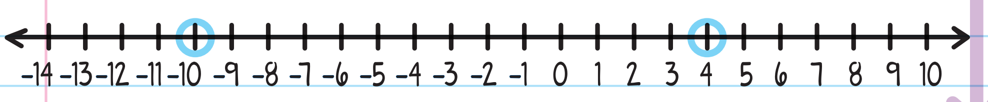
Determine when each of the numerator and denominator is equal to zero.

numerator	denominator
$x + 10 = 0$	$x - 4 = 0$
$x = -10$	$x = 4$

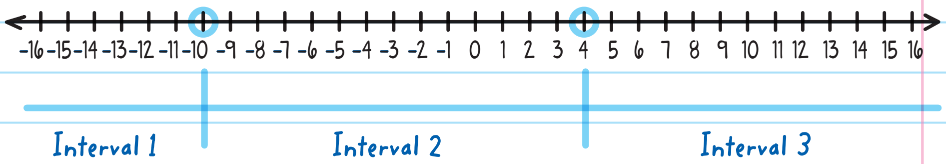
Critical values: $x = -10$ and $x = 4$

Step 4: Plot the critical values on a real number line.

NOTE: Critical values that make a denominator zero will ALWAYS be drawn with an open circle.

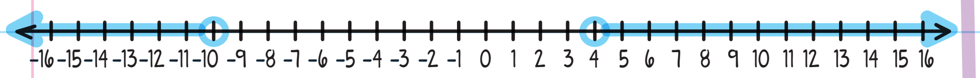


Step 5: Test a value in each interval.



INTERVAL 1	INTERVAL 2	INTERVAL 3
$\frac{3x+2}{x-4} > 2$	$\frac{3x+2}{x-4} > 2$	$\frac{3x+2}{x-4} > 2$
Test $x = -12$	Test $x = 0$	Test $x = 5$
$\frac{3(-12)+2}{-12-4} > 2$	$\frac{3(0)+2}{0-4} > 2$	$\frac{3(5)+2}{5-4} > 2$
$\frac{-34}{-16} > 2$	$\frac{2}{-4} > 2$	$\frac{17}{1} > 2$
$\frac{17}{8} > 2$ ✓	$-\frac{1}{2} > 2$ ✗	$17 > 2$ ✓
All real numbers in this interval satisfy the inequality.	All real numbers in this interval do NOT satisfy the inequality.	All real numbers in this interval satisfy the inequality.

Step 6: Shade the number line. Then write the solution.



Solution set in inequality notation: $x < -10$ or $x > 4$

Solution set in interval notation: $(-\infty, -10) \cup (4, \infty)$

So, the solution set for the rational inequality $\frac{3x+2}{x-4} > 2$ is $(-\infty, -10) \cup (4, \infty)$.

EXAMPLE: Solve $\frac{x^2-3x-4}{x-3} \leq 0$.

Step 1: Rewrite the inequality as an equation and factor the numerator.

$$\frac{(x+1)(x-4)}{x-3} = 0$$

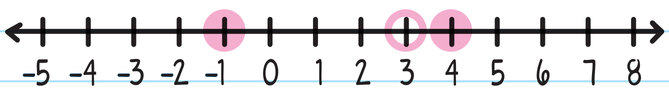
Step 2: Find the critical values.

numerator		denominator
$x + 1 = 0$	$x - 4 = 0$	$x - 3 = 0$
$x = -1$	$x = 4$	$x = 3$

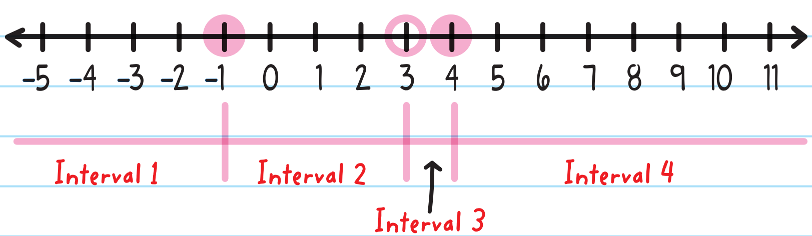
Critical values: $x = -1$, $x = 4$, and $x = 3$

Step 3: Plot the critical values on a real number line.

Since the expression is *greater than or equal to* 0, plot −1 and 4 using a *closed circle*. Plot 3 with an *open circle* because it makes the denominator 0.

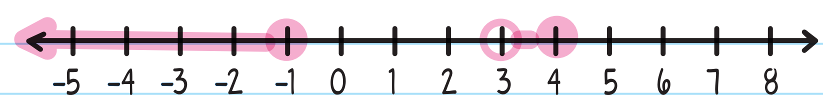


Step 4: Test a value in each interval.



INTERVAL 1	INTERVAL 2	INTERVAL 3	INTERVAL 4
$\frac{x^2-3x-4}{x-3} \leq 0$	$\frac{x^2-3x-4}{x-3} \leq 0$	$\frac{x^2-3x-4}{x-3} \leq 0$	$\frac{x^2-3x-4}{x-3} \leq 0$
Test $x = -2$	Test $x = 0$	Test $x = 3.5$	Test $x = 6$
$\frac{(-2)^2-3(-2)-4}{-2-3} \leq 0$	$\frac{0^2-3(0)-4}{0-3} \leq 0$	$\frac{3.5^2-3(3.5)-4}{3.5-3} \leq 0$	$\frac{6^2-3(6)-4}{6-3} \leq 0$
$\frac{6}{-5} \leq 0$	$\frac{-4}{-3} \leq 0$	$\frac{-2.25}{0.5} \leq 0$	
$-\frac{6}{5} \leq 0$ ✓	$\frac{4}{3} \leq 0$ ✗	$-4.5 \leq 0$ ✓	$\frac{14}{3} \leq 0$ ✗
All real numbers in this interval satisfy the inequality.	Real numbers in this interval do not satisfy the inequality.	All real numbers in this interval satisfy the inequality.	Real numbers in this interval do not satisfy the inequality.

Step 5: Shade the number line. Then write the solution set.



Solution set in inequality notation: $x \leq -1$ or $3 < x \leq 4$

Solution set in interval notation: $(-\infty, -1] \cup (3, 4]$

So, the solution set for the rational inequality $\frac{x^2 - 3x - 4}{x - 3} \le 0$ is $(-\infty, -1] \cup (3, 4]$.

CHECK YOUR KNOWLEDGE

For exercises 1 through 3, solve the rational equation.

1. $\frac{x+4}{x+6} - \frac{x+1}{x+2} = 0$

2. $\frac{4}{x+2} = \frac{6}{x+4} - \frac{12}{x^2+6x+8}$

3. $\frac{2x+2}{x+7} = \frac{x+1}{x+2}$

For exercises 4 through 6, solve the rational inequalities.

4. $\frac{x^2 - 2x - 15}{x - 2} \ge 0$

5. $\frac{x^2 - 2x}{(x+5)(x-2)} \le 0$

6. $\frac{x^2 - x - 6}{(x-1)^2} > 0$

ANSWERS

CHECK YOUR ANSWERS

1. $x = 2$

2. $x = 8$

3. $x = -1$ and $x = 3$

4. $[-3, 2) \cup [5, \infty]$

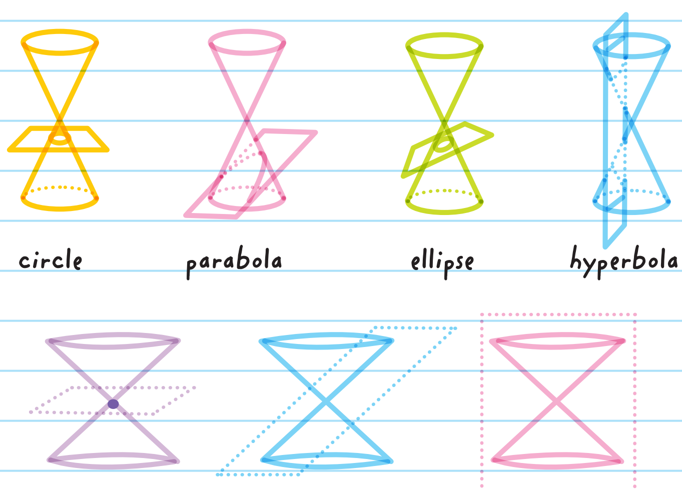
5. $(-5, 0]$

6. $(-\infty, -2) \cup (3, \infty)$

CONIC SECTIONS

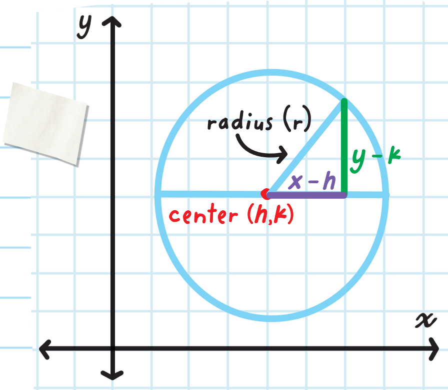
A **CONIC SECTION** is the intersection of a plane and a right circular cone.

There are different types of conic sections:



EQUATIONS OF CIRCLES

A **CIRCLE** is the set of all points at a fixed distance from a fixed point. The fixed distance is called the **RADIUS** of the circle. The fixed point is called the **CENTER** of the circle.



We can find an equation of a circle with center (h, k) and radius r by using the **DISTANCE FORMULA**.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

First, substitute (h, k) for (x_1, y_1) and r for d into the distance formula:

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

NOTE: There is no reason to include the subscripts on the variables x and y anymore.

Then square each side of the equation and simplify.

$$r^2 = \left(\sqrt{(x-h)^2 + (y-k)^2}\right)^2$$

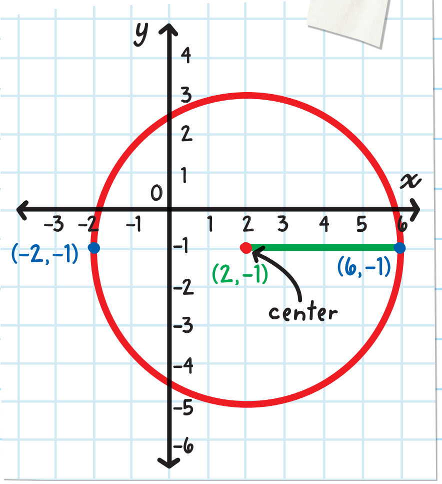
$$r^2 = (x-h)^2 + (y-k)^2$$

This is the standard form of the equation of a circle.

The standard form of the **EQUATION OF A CIRCLE** with center (h, k) and radius r is:

$$(x - h)^2 + (y - k)^2 = r^2.$$

EXAMPLE: Find the standard form of the equation of the circle shown in the graph.



Step 1: Identify the center and radius of the circle.

Center: $(h, k) = (2, -1)$;
radius: $r = 6 - 2 = 4$

Step 2: Substitute h, k, and r into the standard form of the equation of a circle.

$(x - h)^2 + (y - k)^2 = r^2$
$(x - 2)^2 + (y - (-1))^2 = 4^2$
$(x - 2)^2 + (y + 1)^2 = 16$

So, the standard form of the equation of the circle is $(x - 2)^2 + (y + 1)^2 = 16$.

Graphing a circle whose equation is NOT in standard form requires a few more steps.

We often use the procedure of **COMPLETING THE SQUARE** to rewrite an equation of a circle in standard form. Once the equation is in standard form, the center and radius can be found.

EXAMPLE: Rewrite the following equation of a circle in standard form and indicate its center and radius. Then graph the circle on the coordinate plane.

Equation of circle: $x^2 + y^2 - 6x - 4y + 9 = 0$

To rewrite the equation in standard form, we will complete the square.

Step 1: Move the constant to the right side of the equation, group the x-terms together, and group the y-terms together.

$x^2 + y^2 - 6x - 4y + 9 - 9 = 0 - 9$

$(x^2 - 6x) + (y^2 - 4y) = -9$

Step 2: Complete the square for each variable and balance the equation.

$(x^2 - 6x) + (y^2 - 4y) = -9$ — Identify the coefficients of the x and y terms.

half of -6 squared — half of -4 squared

$(x^2 - 6x + 9) + (y^2 - 4y + 4) = -9 + 9 + 4$ — Square each coefficient, divide the result by 2, and add that value.

Add the same values to the other side to balance the equation.

$(x^2 - 6x + 9) + (y^2 - 4y + 4) = 4$ — Now the left-hand side of the equation is the sum of two perfect squares.

Step 3: Factor the perfect squares.

$(x^2 - 6x + 9) + (y^2 - 4y + 4) = 4$

becomes $(x - 3)^2 + (y - 2)^2 = 4$

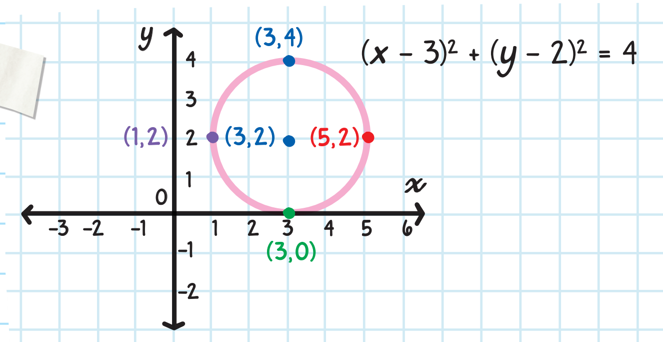
Step 4: Locate the center of the circle and its radius.

The center is $(h, k) = (3, 2)$ and the radius is $r = 2$.

Think: $2^2 = 4$

Step 5: Draw the graph of the circle.

Plot the center. Plot four additional points using the radius. Connect the points to form the circle.



EQUATIONS OF ELLIPSES

An **ELLIPSE** is a stretched out circle. In this chapter, the stretching will occur either horizontally or vertically.

The standard form of the **EQUATION OF AN ELLIPSE** with center (h, k) is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

In this equation:

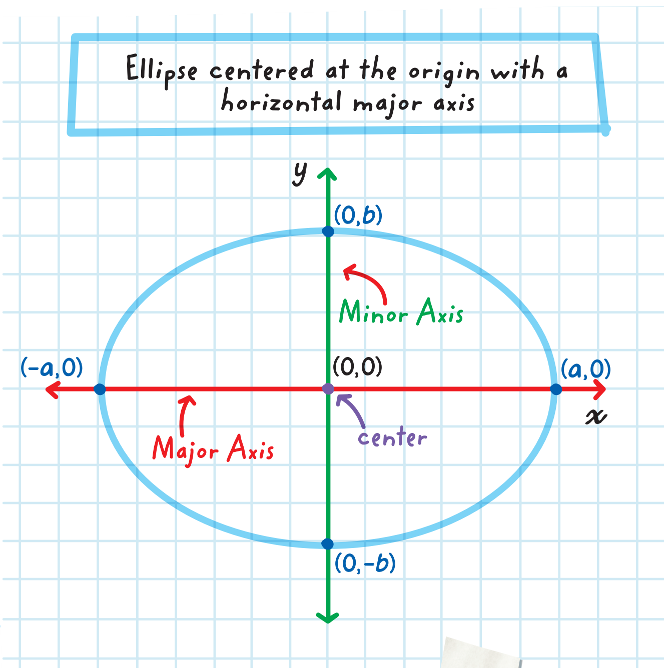
- a is the horizontal distance from the center to the rightmost or leftmost point on the ellipse. (These two points are called **vertices**.)

- b is the vertical distance from the center to the top or bottom point on the ellipse. (These two points are called **vertices**.)

The **MAJOR AXIS** is the line segment between the two vertices of the ellipse that are farthest from each other.

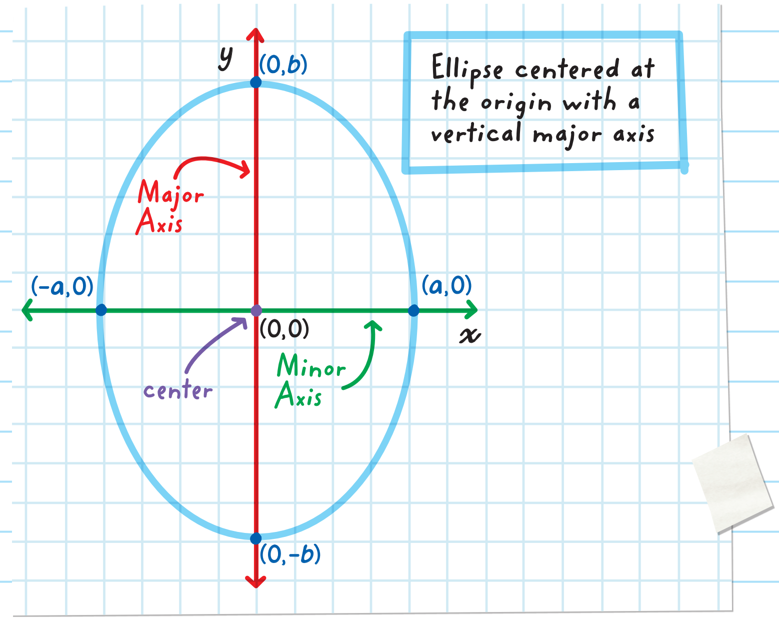
The **MINOR AXIS** is the line segment between the other two vertices of the ellipse.

If $a > b$, then the ellipse will have a *horizontal* major axis. In this case, the length of the major axis will be $2a$ and the length of the minor axis will be $2b$.

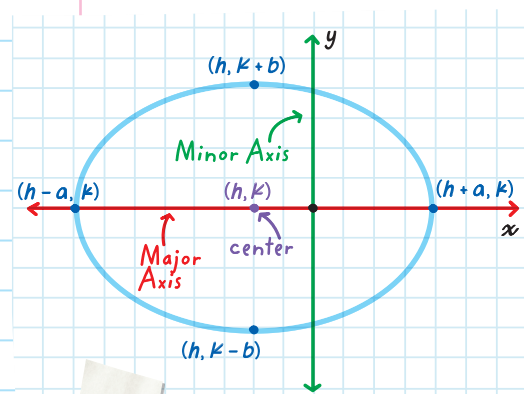


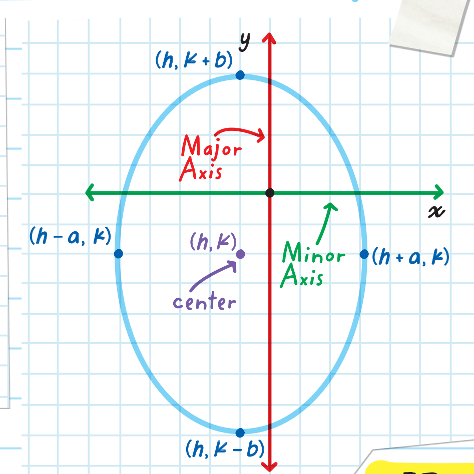
The four points labeled on the graph are the vertices of the ellipse.

If $a < b$, then the ellipse will have a *vertical* major axis. In this case, the length of the major axis will be $2b$ and the length of the minor axis will be $2a$.

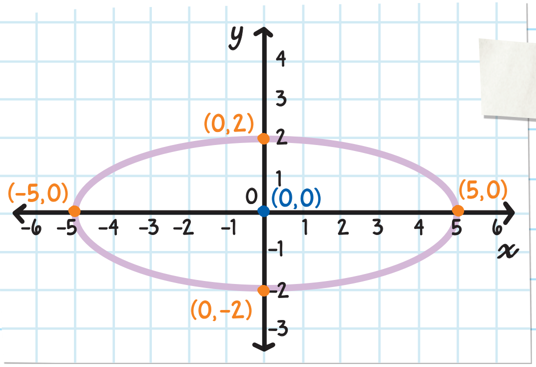


Ellipses are NOT always centered at the origin.





EXAMPLE: Find the standard form of the equation of the ellipse shown in the graph.



Step 1: Identify the center (h, k) and the lengths of the major and minor axes ($2a$ and $2b$) from the graph.

This ellipse has a horizontal major axis with its center at the origin.

center: $(h, k) = (0, 0)$

The length of the major axis is $2a = 5 - (-5) = 10$.

So, $a = 5$.

x-value of right vertex − x-value of left vertex = total length of axis

The length of the minor axis is $2b = 2 - (-2) = 4$.

So, $b = 2$.

y-value of top vertex − y-value of bottom vertex = total length of axis

Step 2: Substitute h, k, a, and b into the standard form of the equation of an ellipse.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-0)^2}{5^2} + \frac{(y-0)^2}{2^2} = 1$$

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

So, the standard form of the equation of the ellipse is $\frac{x^2}{25} + \frac{y^2}{4} = 1$.

EXAMPLE: Rewrite the following equation of an ellipse in standard form and indicate its center and vertices. Then graph the ellipse on the coordinate plane.

$$9x^2 + 4y^2 - 54x + 8y + 49 = 0$$

To rewrite the equation, we will complete the square.

Step 1: Move the constant to the right side of the equation, group the x-terms together, and group the y-terms together. Then factor each variable group.

$9x^2 + 4y^2 - 54x + 8y + 49 - 49 = 0 - 49$ Move the constant.

$(9x^2 - 54x) + (4y^2 + 8y) = -49$ Group the terms and factor.

$9(x^2 - 6x) + 4(y^2 + 2y) = -49$

Step 2: Complete the square for both variables and balance the equation.

$9(x^2 - 6x) + 4(y^2 + 2y) = -49$

$\left(-\frac{6}{2}\right)^2$ $\left(\frac{2}{2}\right)^2$

$9(x^2 - 6x + 9) + 4(y^2 + 2y + 1) = -49 + 81 + 4$

Think: We are adding 9(9) = 81 and 4(1) = 4 to the left side of the equation. So, we need to add 81 and 4 to the right side.

$9(x^2 - 6x + 9) + 4(y^2 + 2y + 1) = 36$

Step 3: Factor the perfect squares and rewrite in standard form.

$9(x - 3)^2 + 4(y + 1)^2 = 36$

$$\frac{9(x-3)^2 + 4(y+1)^2}{36} = \frac{36}{36}$$

$$\frac{9(x-3)^2}{36} + \frac{4(y+1)^2}{36} = 1$$

$$\frac{(x-3)^2}{4} + \frac{(y+1)^2}{9} = 1$$

Step 4: Locate the center of the ellipse and its four vertices.

The center is $(h, k) = (3, -1)$.

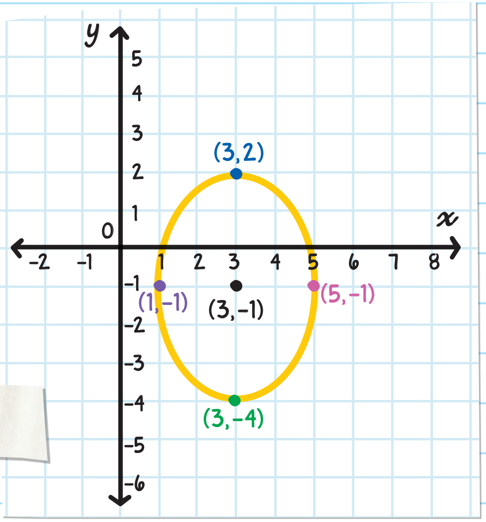
Think: Since $a < b$, the ellipse has a vertical major axis.

Use a and b to find the four vertices.

Since $a = \sqrt{4} = 2$, there are vertices at $(3 - 2, -1) = (1, -1)$ and at $(3 + 2, -1) = (5, -1)$.

Since $b = \sqrt{9} = 3$, there are vertices at $(3, -1 - 3) = (3, -4)$ and at $(3, -1 + 3) = (3, 2)$.

Step 5: Plot the vertices and connect the points to graph the ellipse.



EQUATIONS OF HYPERBOLAS

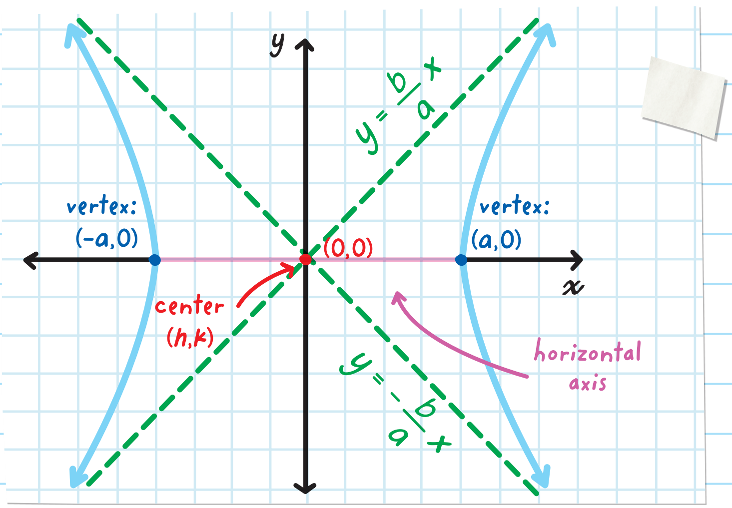
A **HYPERBOLA** is like an ellipse that has been turned inside out. They can have a horizontal axis or a vertical axis.

The standard form of the **EQUATION OF A HYPERBOLA** with a *horizontal axis* and center (h, k) is:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

In this equation: a is the horizontal distance from the center to the nearest point on the left or right. (These two points are called **VERTICES**.)

The **HORIZONTAL AXIS** is the horizontal line segment between the two vertices of the hyperbola. The length of the horizontal axis is $2a$.



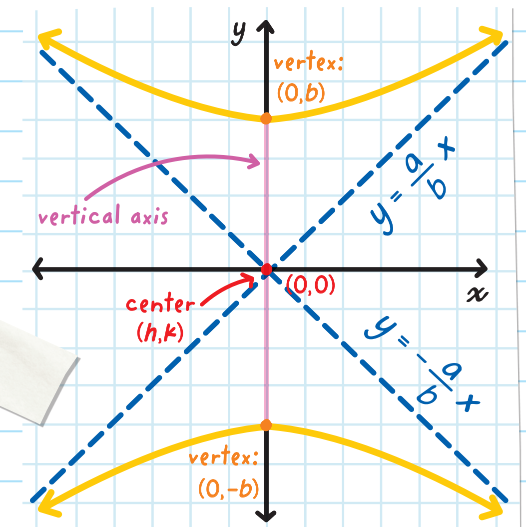
Hyperbola with a horizontal axis

The standard form of the **EQUATION OF A HYPERBOLA** with a *vertical axis* and center (h, k) is:

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

In this equation: b is the vertical distance from the center to the nearest point above and below. (These two points are called vertices.)

The **VERTICAL AXIS** is the vertical line segment between the two vertices of the hyperbola. The length of the vertical axis is $2b$.



Hyperbola with a vertical axis

The **ASYMPTOTES** of a hyperbola are straight lines that guide its shape as it goes to infinity.

For a hyperbola with a horizontal axis, the equations of the asymptotes is:

$$y = k \pm \frac{b}{a}(x - h)$$

For a hyperbola with a vertical axis, the equations of the asymptotes is:

$$y = k \pm \frac{a}{b}(x - h)$$

Note: The difference between the asymptote equations is that a and b are flipped.

EXAMPLE: Graph the hyperbola $\frac{(y-1)^2}{16} - \frac{(x+2)^2}{16} = 1$.

Step 1: Identify h, k, a, and b.

From the given equation, which is in standard form, we see that $h = -2$, $k = 1$, $a = \sqrt{16} = 4$, and $b = \sqrt{16} = 4$.

Step 2: Plot the center (−2, 1).

Step 3: Use b to find the two vertices and plot them.

Since $b = 4$, there are vertices at $(-2, 1 - 4) = (-2, -3)$ and at $(-2, 1 + 4) = (-2, 5)$.

Step 4: Find the asymptotes.

$$y = k \pm \frac{a}{b}(x - h)$$

$$y = 1 \pm \frac{4}{4}(x - (-2))$$

$$y = 1 \pm (x + 2)$$

$$y = 1 + (x + 2) \qquad y = 1 - (x + 2)$$

$$y = x + 3 \qquad y = -x - 1$$

Step 5 (optional): Find one or two more points on the hyperbola.

We can find what the y-values are for $x = 1$ by substituting 1 for x in the original formula.

$$\frac{(y-1)^2}{16} - \frac{(1+2)^2}{16} = 1$$

$$16\left[\frac{(y-1)^2}{16} - \frac{(1+2)^2}{16}\right] = 1(16)$$

$$(y-1)^2 - 3^2 = 16$$

$$(y-1)^2 - 9 = 16$$

$(y-1)^2 - 9 + 9 = 16 + 9$

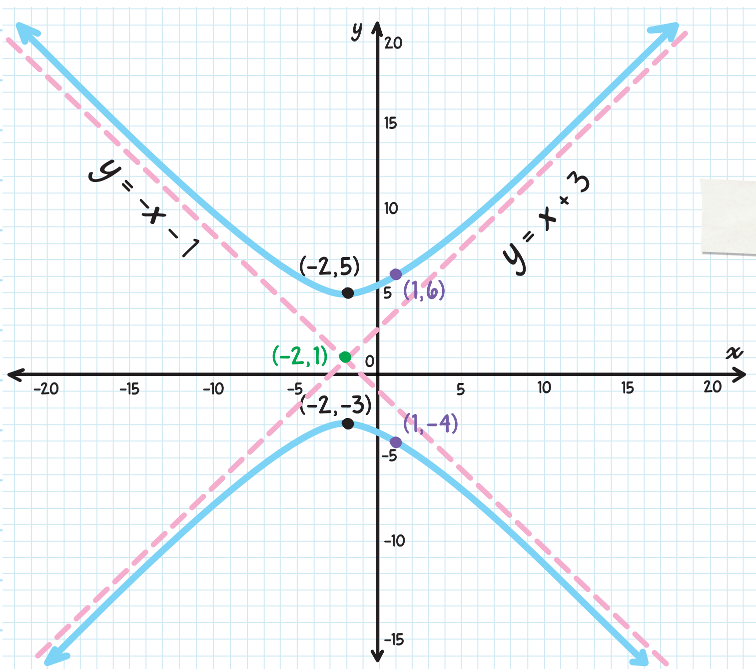
$(y-1)^2 = 25$

$y - 1 = \pm 5$

$y - 1 = 5$	$y - 1 = -5$
$y = 6$	$y = -4$

So, two points on the graph of the hyperbola are (1, 6) and (1, −4).

Step 6: Draw a hyperbola through the plotted points.



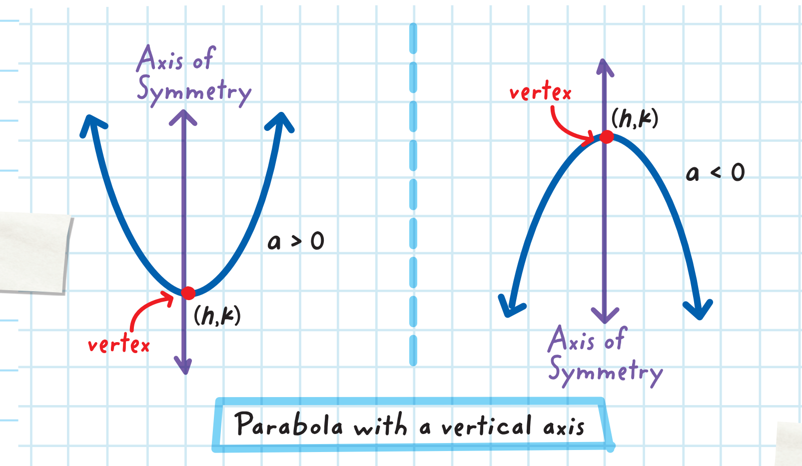
The graph shown is the hyperbola with equation

$$\frac{(y-1)^2}{16} - \frac{(x+2)^2}{16} = 1.$$

EQUATIONS OF PARABOLAS

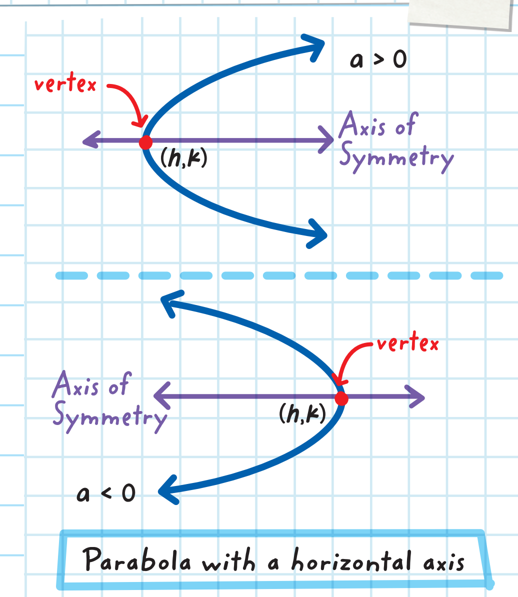
The standard form of the **EQUATION OF A PARABOLA** with a *vertical axis of symmetry* and vertex (h, k) is:

$$(x - h)^2 = a(y - k).$$

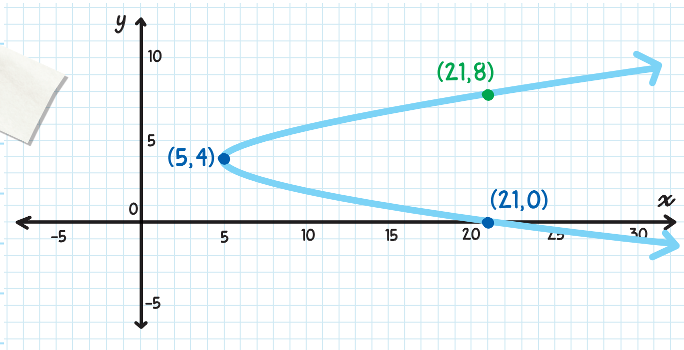


The standard form of the **EQUATION OF A PARABOLA** with a *horizontal axis of symmetry* and vertex (h, k) is:

$$(y - k)^2 = a(x - h).$$



EXAMPLE: Find the standard form of the equation of the parabola shown in the graph.



Step 1: Identify the vertex.

vertex: $(h, k) = (5, 4)$

Step 2: Substitute h and k into the standard form of the equation of a parabola with a horizontal axis.

$(y - k)^2 = a(x - h)$ standard form of the equation of a parabola with a horizontal axis

$(y - 4)^2 = a(x - 5)$

Step 3: To find the value of a, substitute one point on the parabola into the equation and solve for a.

Point: (21, 0)

$(y - 4)^2 = a(x - 5)$

$(0 - 4)^2 = a(21 - 5)$

$(-4)^2 = a(16)$

$16 = 16a$

$16 \div 16 = 16a \div 16$

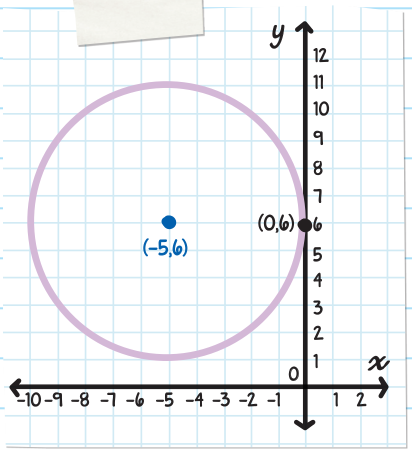
$1 = a$

Equation: $(y - 4)^2 = 1(x - 5)$

So, the equation of the given parabola in standard form is $(y - 4)^2 = x - 5$.

CHECK YOUR KNOWLEDGE

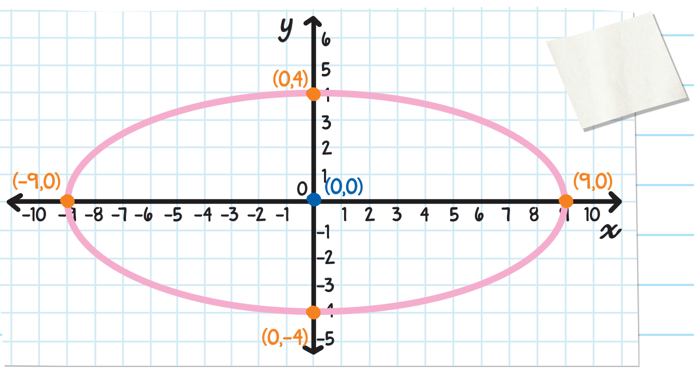
1. Find the standard form of the equation of the circle shown in the graph. Indicate its center and radius.



2. Rewrite the following equation of a circle in standard form and indicate its center and radius. Then graph the circle on the coordinate plane.

$$x^2 + y^2 + 8x - 6y - 11 = 0$$

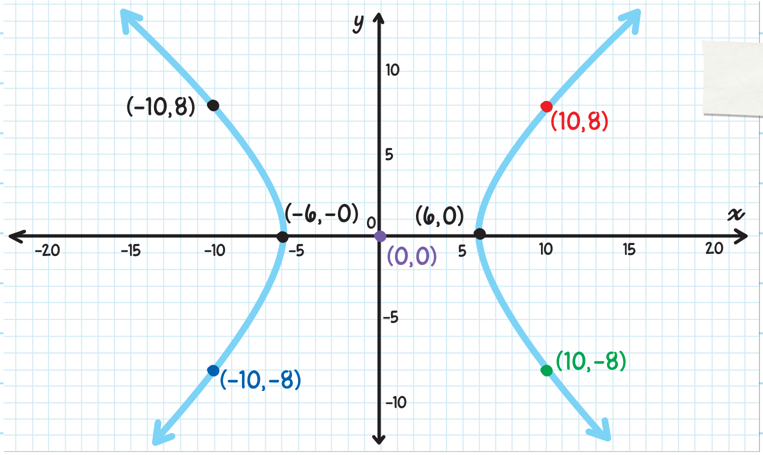
3. Find the standard form of the equation of the ellipse shown in the graph. Indicate its center and vertices.



4. Rewrite the following equation of an ellipse in standard form and indicate its center and vertices. Then graph the ellipse on the coordinate plane.

$4x^2 + y^2 - 32x - 14y + 49 = 0$

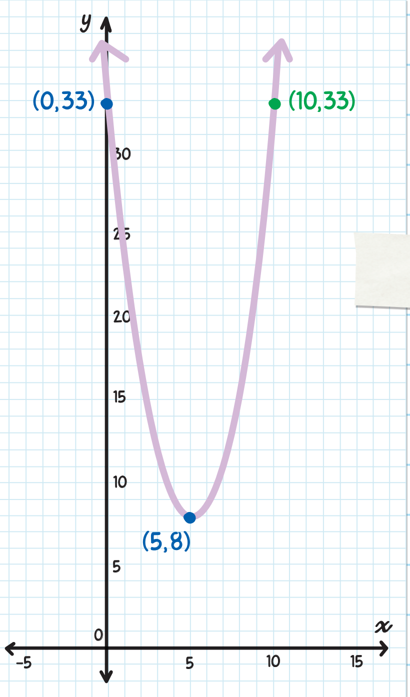
5. Find the standard form of the equation of the hyperbola shown in the graph. Indicate its center and vertices.



6. Rewrite the following equation of a hyperbola in standard form and indicate its center and vertices.

$9x^2 - 16y^2 - 36x - 32y - 124 = 0$

7. Find the standard form of the equation of the parabola shown in the graph. Indicate its vertex and axis.



8. Rewrite the following equation of a parabola in standard form and indicate its vertex and axis.

$y^2 - x + 8y - 20 = 0$

ANSWERS

CHECK YOUR ANSWERS

1. $(x + 5)^2 + (y - 6)^2 = 25$

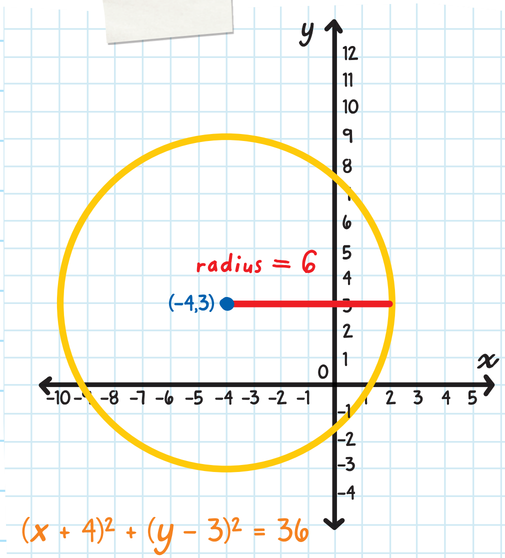
center: $(-5, 6)$

radius: 5

2. $(x + 4)^2 + (y - 3)^2 = 36$

center: $(-4, 3)$

radius: 6



3. $\frac{x^2}{81} + \frac{y^2}{16} = 1$

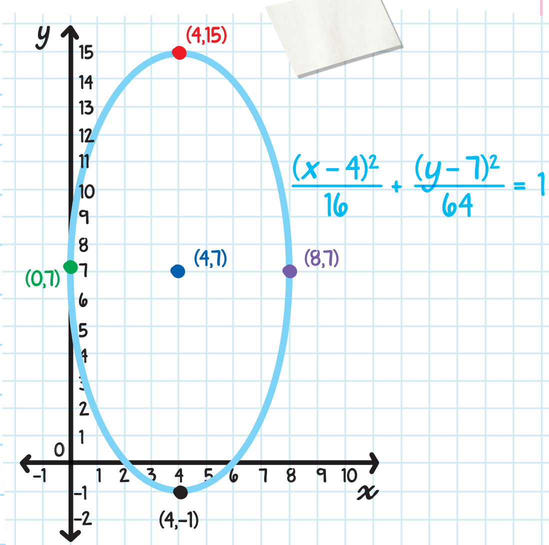
center: $(0, 0)$

vertices: $(0, 4)$, $(0, -4)$, $(9, 0)$, $(-9, 0)$

4. $\frac{(x-4)^2}{16} + \frac{(y-7)^2}{64} = 1$

center: $(4, 7)$

vertices: $(0, 7)$, $(8, 7)$, $(4, 15)$, $(4, -1)$



5. $\frac{x^2}{36} - \frac{y^2}{36} = 1$

center: $(0, 0)$

vertices: $(-6, 0)$, $(6, 0)$

6. $\frac{(x-2)^2}{16} - \frac{(y+1)^2}{9} = 1$

center: $(2, -1)$

vertices: $(-2, -1)$, $(6, -1)$

7. $(x - 5)^2 = y - 8$

vertex: $(5, 8)$

axis: $x = 5$

8. $x + 36 = (y + 4)^2$

vertex: $(-36, -4)$

axis: $y = -4$

MATRIX OPERATIONS

A **MATRIX** is a rectangular array of numbers.

Here are some examples:

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & -2 & 2 \\ -1 & 4 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -5 & 3 & -1 \\ -4 & 0 & 2 & -2 \\ 0 & -2 & -4 & 1 \end{bmatrix}$$

A is a 2×2 **MATRIX** because it has 2 rows and 2 columns.

2 columns

2 rows $\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$

B is a 2×3 **MATRIX**. **C** is a 3×4 **MATRIX**.

3 columns

2 rows $\begin{bmatrix} -3 & -2 & 2 \\ -1 & 4 & 5 \end{bmatrix}$

4 columns

$\begin{bmatrix} 2 & -5 & 3 & -1 \\ -4 & 0 & 2 & -2 \\ 0 & -2 & -4 & 1 \end{bmatrix}$ 3 rows

Two matrices are **EQUAL** if they have the same size and all their elements are equal.

For example, if $\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$,

then $x = 0$, $y = 1$, $z = 3$, and $w = 2$.

ADDITION AND SUBTRACTION OF MATRICES

We can add or subtract two matrices *only* when they have the same size.

For example, we can add the matrices below because both have the same size: 3×2.

Add corresponding elements.

$$\begin{bmatrix} 1 & -2 \\ 2 & 3 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 1+2 & -2+1 \\ 2+3 & 3+0 \\ 0+5 & -1+(-2) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 5 & 3 \\ 5 & -3 \end{bmatrix}$$

We can subtract the matrices below because both have the same size: 3×2.

Subtract corresponding elements.

$$\begin{bmatrix} 1 & -2 \\ 2 & 3 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 1-2 & -2-1 \\ 2-3 & 3-0 \\ 0-5 & -1-(-2) \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ -1 & 3 \\ -5 & 1 \end{bmatrix}$$

MULTIPLICATION OF MATRICES

To multiply a matrix by a real number (called a **SCALAR**), multiply each element by that number.

EXAMPLE 1:

$$3\begin{bmatrix} 2 & -5 & 3 \\ 1 & -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \cdot 2 & 3 \cdot (-5) & 3 \cdot 3 \\ 3 \cdot 1 & 3 \cdot (-1) & 3 \cdot (-2) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -15 & 9 \\ 3 & -3 & -6 \end{bmatrix}$$

EXAMPLE 2:

$$2\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$$

This problem has multiplication and subtraction.

$$= \begin{bmatrix} 2 \cdot 2 & 2 \cdot 3 \\ 2 \cdot 5 & 2 \cdot (-2) \end{bmatrix} - \begin{bmatrix} 5 \cdot 1 & 5 \cdot 0 \\ 5 \cdot 2 & 5 \cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 \\ 10 & -4 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 10 & 10 \end{bmatrix}$$

Subtract corresponding elements.

$$= \begin{bmatrix} -1 & 6 \\ 0 & -14 \end{bmatrix}$$

We can multiply two matrices together if the **number of columns** of the *first matrix* is **equal** to the **number of rows** of the *second matrix*.

For example, consider matrices **A**, **B**, and **C** below:

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 \\ 2 & 3 \\ 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \end{bmatrix}$$

2×2 $\quad$ 3×2 $\quad$ 2×3

We can multiply **A** times **C** because **A** has 2 columns and **C** has 2 rows.

2 columns

$$\mathbf{AC} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \end{bmatrix}$$

2 rows

We can multiply **B** times **C** because **B** has 2 columns and **C** has 2 rows.

2 columns

$$\mathbf{BC} = \begin{bmatrix} 1 & -2 \\ 2 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \end{bmatrix}$$

2 rows

We *cannot* multiply **A** times **B** because **A** has 2 columns and **B** has 3 rows.

2 columns

$$\mathbf{AB} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 3 \\ 0 & -1 \end{bmatrix}$$

3 rows

These are the products we can form:

AA **AC** **BA** **BC** **CB**

These here are the products we cannot form:

AB **BB** **CA** **CC**

So, how do we multiply two matrices?

For each row of the first matrix and each column of the second matrix, we add up the products element by element.

Let's compute the product **AC** as an example.

$$\mathbf{AC} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}$$

2×2 2×3 2×3

Since x is in the first row and first column, multiply the first numbers in the first row of **A** and the first column of **C**. Then multiply the second numbers. Add the products.

$$x = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \cdot 1 + 1 \cdot 0 = 0 + 0 = 0.$$

Since u is in the second row and first column, multiply the first numbers in the second row of **A** and the first column of **C**. Then multiply the second numbers. Add the products.

$$u = \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3 \cdot 1 + 2 \cdot 0 = 3 + 0 = 3.$$

Follow this procedure to compute the values of the remaining elements y, z, v, and w.

The final product is

$$AC = \begin{bmatrix} 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 2 + 1 \cdot 3 & 0 \cdot 0 + 1 \cdot 6 \\ 3 \cdot 1 + 2 \cdot 0 & 3 \cdot 2 + 2 \cdot 3 & 3 \cdot 0 + 2 \cdot 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 & 6 \\ 3 & 12 & 12 \end{bmatrix}$$

Notice that the product of a 2×2 matrix and a 2×3 matrix is a 2×3 matrix.

The innermost numbers (both 2) must agree, and the resulting product has a size given by the outermost numbers (2 and 3).

EXAMPLE: Find the product.

First determine if you can multiply these two matrices.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 & 9 \\ 7 & 2 & 9 \\ 3 & 8 & 1 \end{bmatrix}$$

The first matrix is a 3×3, and the second matrix is also a 3×3. Since the number of columns in the first matrix (3) matches the numbers of rows in the second matrix (3), these matrices can be multiplied.

$$= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 7 + 3 \cdot 3 & 1 \cdot 5 + 2 \cdot 2 + 3 \cdot 8 & 1 \cdot 9 + 2 \cdot 9 + 3 \cdot 1 \\ 4 \cdot 1 + 5 \cdot 7 + 6 \cdot 3 & 4 \cdot 5 + 5 \cdot 2 + 6 \cdot 8 & 4 \cdot 9 + 5 \cdot 9 + 6 \cdot 1 \\ 7 \cdot 1 + 8 \cdot 7 + 9 \cdot 3 & 7 \cdot 5 + 8 \cdot 2 + 9 \cdot 8 & 7 \cdot 9 + 8 \cdot 9 + 9 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 33 & 30 \\ 57 & 78 & 87 \\ 90 & 123 & 144 \end{bmatrix}$$

Remember: multiply the rows of the first matrix with the columns of the second matrix.

REAL-WORLD APPLICATION OF MATRICES

Matrices are often used in the real world to organize and make sense of data sets.

Let's look at a retail example.

EXAMPLE: At Bellany's Boutique, classic T-shirts cost \$15.50, lounge pants cost \$21.75, and colored socks cost \$10.99. Gilda purchases 5 classic T-shirts, 3 pairs of lounge pants, and 4 pairs of colored socks. Patricia purchases 4 classic T-shirts, 2 pairs of lounge pants, and 6 pairs of colored socks. Which individual spent the most money?

To solve this problem, we can use matrices to find the amount of money Gilda spent and the amount of money Particia spent. From there, we can determine who spent the most.

Step 1: Set up a matrix **A** to show the number of items each woman purchased. Then set up a matrix **B** to show the cost of each item.

Matrix A:

Let the top row indicate Gilda's purchases.

Let the bottom row indicate Patricia's purchases.

$$A = \begin{bmatrix} 5 & 3 & 4 \\ 4 & 2 & 6 \end{bmatrix} \begin{matrix} \leftarrow \text{Gilda's purchases} \\ \leftarrow \text{Patricia's purchases} \end{matrix}$$

Matrix B:

Let the single column indicate the price of each item.

$$B = \begin{bmatrix} 15.50 \\ 21.75 \\ 10.99 \end{bmatrix}$$

$$\underset{2 \times 3}{\begin{bmatrix} 5 & 3 & 4 \\ 4 & 2 & 6 \end{bmatrix}} \cdot \underset{3 \times 1}{\begin{bmatrix} 15.50 \\ 21.75 \\ 10.99 \end{bmatrix}} = \underset{2 \times 1}{AB}$$

THINK: Since the number of columns in the first matrix (3) matches the numbers of rows in the second matrix (3), we can multiply these matrices.

Step 2: Find the product matrix **AB** to determine the amount of money Gilda spent and the amount of money Patricia spent.

$$AB = \begin{bmatrix} 5 \cdot 15.50 + 3 \cdot 21.75 + 4 \cdot 10.99 \\ 4 \cdot 15.50 + 2 \cdot 21.75 + 6 \cdot 10.99 \end{bmatrix}$$

$$= \begin{bmatrix} 77.50 + 65.25 + 43.96 \\ 62.00 + 43.50 + 65.94 \end{bmatrix}$$

$$= \begin{bmatrix} 186.71 \\ 171.44 \end{bmatrix}$$

Gilda spent $186.71. Patricia spent $171.44. Therefore, Gilda spent more money at Bellany's Boutique.

CHECK YOUR KNOWLEDGE

1. Given the following matrices **A** and **B**, determine whether **A** = **B**. Justify your answer.

$$A = \begin{bmatrix} 2 & -2 \\ 4 & 1 \\ 5 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 2 & -2 \\ -4 & 1 \\ 5 & -1 \end{bmatrix}$$

For questions 2 through 7, perform the indicated operation.

2. $\begin{bmatrix} -2 & 3 \\ 8 & -5 \\ 7 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -2 & 9 \\ 11 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 5 & 14 & 7 \\ -1 & 10 & -6 \\ 6 & 8 & 0 \end{bmatrix} - \begin{bmatrix} 5 & 12 & 15 \\ -1 & 11 & -20 \\ 7 & 18 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 15 & -11 \\ -9 & 1 & -2 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 0 \\ 1 & 8 & -6 \end{bmatrix}$

5. $\begin{bmatrix} 25 & 30 \\ -62 & -45 \end{bmatrix} - \begin{bmatrix} 102 & -50 \\ -31 & 17 \end{bmatrix}$

6. $\begin{bmatrix} 6 & -2 \\ 2 & 2 \\ 0 & -7 \end{bmatrix} \cdot \begin{bmatrix} 2 & -12 \\ -6 & 35 \end{bmatrix}$

7. $\begin{bmatrix} -20 & 11 \\ -8 & 25 \end{bmatrix} \cdot \begin{bmatrix} 24 & -4 \\ -6 & 3 \end{bmatrix}$

8. The manager of a home decor store purchases 75 bedsheet sets for \$46 each and sells 48 pillows for \$28 each. Did this store make a profit or loss from these purchases and sales? Use matrix multiplication to find the total revenue.

9. Explain why matrices **A** and **B** cannot be added. Then explain why they cannot be multiplied.

$$A = \begin{bmatrix} 7 & -1 & 0 \\ -9 & 6 & 14 \end{bmatrix}, \; B = \begin{bmatrix} -5 & 8 \\ 31 & 20 \end{bmatrix}$$

ANSWERS

CHECK YOUR ANSWERS

1. $A \neq B$.
Two matrices are equal if they have the same size and all of their elements are equal. For the given matrices, one element (4) in the first matrix (Row 2, Column 1) does not equal the corresponding element in the second matrix (−4).

2. $\begin{bmatrix} -2 & 2 \\ 6 & 4 \\ 18 & 9 \end{bmatrix}$

3. $\begin{bmatrix} 0 & 2 & -8 \\ 0 & -1 & 14 \\ -1 & -10 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 13 & -11 \\ -8 & 9 & -8 \end{bmatrix}$

5. $\begin{bmatrix} -77 & 80 \\ -31 & -62 \end{bmatrix}$

6. $\begin{bmatrix} 24 & -142 \\ -8 & 46 \\ 42 & -245 \end{bmatrix}$

7. $\begin{bmatrix} -546 & 113 \\ -342 & 107 \end{bmatrix}$

8. $\begin{bmatrix} 75 & 48 \end{bmatrix} \cdot \begin{bmatrix} -46 \\ 28 \end{bmatrix} = \begin{bmatrix} -2{,}106 \end{bmatrix}$

A negative answer means the total revenue represents a loss. The loss was $2,106.

9. To add two matrices, they must be the same size. **A** is a 2×3 matrix and **B** is 2×2 matrix.

To multiply two matrices, the number of columns in the first matrix must be the same as the number of rows in the second matrix. Since **A** has 3 columns and **B** has 2 rows, this is not the case, and therefore **A** and **B** cannot be multiplied.

DETERMINANTS AND CRAMER'S RULE

The **DETERMINANT** of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - cb$.

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

The determinant of the matrix **A** is denoted by *det A* or $|A|$.

For example, let's compute the determinant of the following 2×2 matrix.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = 0 \cdot 2 - 1 \cdot 3 = 0 - 3 = -3$$

The **determinant** of the 3×3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is $(aei + bfg + cdh) - (gec + hfa + idb)$.

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

$$= (aei + bfg + cdh) - (gec + hfa + idb)$$

Here is another way to do the computation.

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= a \cdot \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \cdot \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

EXAMPLE: Evaluate the determinant of each matrix.

1. $\mathbf{A} = \begin{bmatrix} 5 & 9 & -1 \\ -3 & -2 & 0 \\ 2 & 10 & 7 \end{bmatrix}$

$|A| = 5 \cdot (-2) \cdot 7 + 9 \cdot 0 \cdot 2 + (-1) \cdot (-3) \cdot 10 - 2 \cdot (-2) \cdot (-1) - 10 \cdot 0 \cdot 5 - 7 \cdot (-3) \cdot 9$

$= 145$

So, $|A| = \begin{vmatrix} 5 & 9 & -1 \\ -3 & -2 & 0 \\ 2 & 10 & 7 \end{vmatrix} = 145$

2. $\begin{bmatrix} -5 & 0 & -1 \\ 1 & 2 & -1 \\ -3 & 4 & 1 \end{bmatrix}$

$$\det \begin{bmatrix} -5 & 0 & -1 \\ 1 & 2 & -1 \\ -3 & 4 & 1 \end{bmatrix}$$

$$= -5 \cdot \det \begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix} - (0) \cdot \det \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix} + (-1) \cdot \det \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$= -5[2 - (-4)] - 0[1 - 3] - 1[4 - (-6)]$

$= -5(2 + 4) - 0 - 1(4 + 6)$

$= -5(6) - 1(10)$

$= -30 - 10$

$= -40$

So, $\det \begin{bmatrix} -5 & 0 & -1 \\ 1 & 2 & -1 \\ -3 & 4 & 1 \end{bmatrix} = -40.$

Determinants can be used to solve linear systems of equations using a method called **CRAMER'S RULE**.

Consider the following linear system:

$$ax + by = e$$
$$cx + dy = f$$

The **COEFFICIENT MATRIX** of this system is the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

This is just like it sounds: a matrix made up of the equation's coefficients.

According to Cramer's rule, the solution of the system:

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases} \text{ is } x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

THINK: Because the coefficient matrix is found in the denominator, Cramer's rule can be used only if the coefficient matrix has a nonzero determinant.

EXAMPLE: Use Cramer's rule to solve the following linear system of equations.

$8x + 2y = 4$
$-2x + 3y = 13$

Substitute. Calculate the determinants. Then solve for x and y.

$$x = \frac{\begin{vmatrix} 4 & 2 \\ 13 & 3 \end{vmatrix}}{\begin{vmatrix} 8 & 2 \\ -2 & 3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 8 & 4 \\ -2 & 13 \end{vmatrix}}{\begin{vmatrix} 8 & 2 \\ -2 & 3 \end{vmatrix}}$$

$$= \frac{(4 \cdot 3) - (2 \cdot 13)}{(8 \cdot 3) - (2 \cdot -2)} \qquad = \frac{(8 \cdot 13) - (4 \cdot -2)}{(8 \cdot 3) - (-2 \cdot -2)}$$

$$= \frac{-14}{28} \qquad = \frac{112}{28}$$

$$= -\frac{1}{2} \qquad = 4$$

So, the solution of the system is $x = -\frac{1}{2}$ and $y = 4$.

CHECK YOUR KNOWLEDGE

For questions 1 through 3, evaluate the determinant of each matrix.

1. $\begin{bmatrix} -2 & -12 \\ 4 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 4 & -8 & 11 \\ 7 & 10 & -2 \\ 2 & -4 & -5 \end{bmatrix}$

3. $\begin{bmatrix} -9 & 14 & 2 \\ 5 & 3 & 1 \\ -6 & 0 & 4 \end{bmatrix}$

For questions 4 through 6, use Cramer's rule to solve the linear system of equations.

4. $\begin{cases} 6x - 3y = 3 \\ 3x - 2y = 1 \end{cases}$

5. $\begin{cases} 2x + y = 9 \\ -2x + 3y = 11 \end{cases}$

6. $\begin{cases} 3x + 2y = 4 \\ 4x + y = -3 \end{cases}$

ANSWERS

CHECK YOUR ANSWERS

1. 36

2. −1,008

3. −436

4. $x = 1$, $y = 1$

5. $x = 2$, $y = 5$

6. $x = -2$, $y = 5$

INVERSES OF MATRICES

The multiplicative identity for the set of real numbers is 1 because $1 \cdot x = x \cdot 1 = x$ for every real number x.

Certain square matrices are also identities. For example, the 2×2 **IDENTITY MATRIX** is the matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This matrix **I** is called an **identity matrix** because whenever we multiply it by another matrix **A**, if the product exists, then the product is equal to **A**.

Similarly, the 3×3 identity matrix is $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

For example, if $A = \begin{bmatrix} 2 & -3 & 4 \\ -6 & 5 & 8 \\ 7 & 1 & 0 \end{bmatrix}$, then $IA = AI = A$.

$$\begin{bmatrix} 2 & -3 & 4 \\ -6 & 5 & 8 \\ 7 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 1 + (-3) \cdot 0 + 4 \cdot 0 & 2 \cdot 0 + (-3) \cdot 1 + 4 \cdot 0 & 2 \cdot 0 + (-3) \cdot 0 + 4 \cdot 1 \\ (-6) \cdot 1 + 5 \cdot 0 + 8 \cdot 0 & (-6) \cdot 0 + 5 \cdot 1 + 8 \cdot 0 & (-6) \cdot 0 + 5 \cdot 0 + 8 \cdot 1 \\ 7 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 & 7 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 & 7 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 & 4 \\ -6 & 5 & 8 \\ 7 & 1 & 0 \end{bmatrix} = A$$ ✓ True

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 & 4 \\ -6 & 5 & 8 \\ 7 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 2 + 0 \cdot (-6) + 0 \cdot 7 & 1 \cdot (-3) + 0 \cdot 5 + 0 \cdot 1 & 1 \cdot 4 + 0 \cdot 8 + 0 \cdot 0 \\ 0 \cdot 2 + 1 \cdot (-6) + 0 \cdot 7 & 0 \cdot (-3) + 1 \cdot 5 + 0 \cdot 1 & 0 \cdot 4 + 1 \cdot 8 + 0 \cdot 0 \\ 0 \cdot 2 + 0 \cdot (-6) + 1 \cdot 7 & 0 \cdot (-3) + 0 \cdot 5 + 1 \cdot 1 & 0 \cdot 4 + 0 \cdot 8 + 1 \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 & 4 \\ -6 & 5 & 8 \\ 7 & 1 & 0 \end{bmatrix} = A$$ ✓ True

The **inverse** of the 2 × 2 matrix **A** = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if it exists, is

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Note: In order for the inverse to exist, the determinant $|A| = ad - bcc$ cannot be 0.

The product of a matrix and its inverse is the identify matrix.

EXAMPLE: Find the inverse of $A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Step 1: Substitute $a = 4$, $b = 3$, $c = 3$, and $d = 2$ in the formula for the inverse.

$$A^{-1} = \frac{1}{4 \cdot 2 - 3 \cdot 3}\begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} = -1\begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$$

Step 2: Confirm that $AA^{-1} = I = A^{-1}A$.

$$\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 4 \cdot (-2) + 3 \cdot 3 & 4 \cdot 3 + 3 \cdot (-4) \\ 3 \cdot (-2) + 2 \cdot 3 & 3 \cdot 3 + 2 \cdot (-4) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ✓$$

$$\begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 \cdot 4 + 3 \cdot 3 & -2 \cdot 3 + 3 \cdot 2 \\ 3 \cdot 4 + (-4) \cdot 3 & 3 \cdot 3 + (-4) \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ✓$$

So, $A^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$.

EXAMPLE: Determine whether the given matrices are inverses of each other.

$$A = \begin{bmatrix} 3 & -4 & 2 \\ 0 & 2 & 1 \\ 4 & -5 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 11 & 2 & -8 \\ 4 & 1 & -3 \\ -8 & -1 & 6 \end{bmatrix}$$

Remember: The product of inverse matrices is the identity matrix.

$$\begin{bmatrix} 3 & -4 & 2 \\ 0 & 2 & 1 \\ 4 & -5 & 3 \end{bmatrix} \cdot \begin{bmatrix} 11 & 2 & -8 \\ 4 & 1 & -3 \\ -8 & -1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot 11+(-4)\cdot 4+2(-8) & 3\cdot 2+(-4)\cdot 1+2(-1) & 3(-8)+(-4)(-3)+2\cdot 6 \\ 0\cdot 11+2\cdot 4+1\cdot(-8) & 0\cdot 2+2\cdot 1+1\cdot(-1) & 0\cdot(-8)+2(-3)+1\cdot 6 \\ 4\cdot 11+(-5)\cdot 4+3(-8) & 4\cdot 2+(-5)\cdot 1+3(-1) & 4(-8)+(-5)(-3)+3\cdot 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ✓ the identity matrix

$$\begin{bmatrix} 11 & 2 & -8 \\ 4 & 1 & -3 \\ -8 & -1 & 6 \end{bmatrix} \cdot \begin{bmatrix} 3 & -4 & 2 \\ 0 & 2 & 1 \\ 4 & -5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11\cdot 3+2\cdot 0+(-8)\cdot 4 & 11(-4)+2\cdot 2+(-8)(-5) & 11\cdot 2+2\cdot 1+(-8)\cdot 3 \\ 4\cdot 3+1\cdot 0+(-3)\cdot 4 & 4(-4)+1\cdot 2+(-3)(-5) & 4\cdot 2+1\cdot 1+(-3)\cdot 3 \\ (-8)\cdot 3+(-1)\cdot 0+6\cdot 4 & (-8)(-4)+(-1)\cdot 2+6(-5) & (-8)\cdot 2+(-1)\cdot 1+6\cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ✓ the identity matrix

Since AB and BA are both equal to the identity matrix, A and B are inverses of each other.

A matrix that has an inverse is called **invertible**, while a matrix that does *not* have an inverse is called **noninvertible** or **singular**.

EXAMPLE: Determine if the given matrix A is invertible.

$$A = \begin{bmatrix} 9 & 6 \\ 12 & 8 \end{bmatrix}$$

A is invertible only if its determinant is not 0.

$$|A| = 9 \cdot 8 - 6 \cdot 12 = 72 - 72 = 0$$

Since $|A| = 0$, A is noninvertible.

CHECK YOUR KNOWLEDGE

For questions 1 through 3, find the inverse of each matrix.

1. $\begin{bmatrix} 5 & -2 \\ -4 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -2 \\ 5 & -1 \end{bmatrix}$

3. $\begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{5}{4} & -1 \end{bmatrix}$

4. Determine if the given matrices are multiplicative inverses of each other.

$$\begin{bmatrix} 3 & 4 & 5 \\ -2 & 5 & 0 \\ 1 & 2 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 10 & 2 & -25 \\ 4 & 1 & -10 \\ -9 & -2 & 23 \end{bmatrix}$$

For questions 5 and 6, determine if the given matrix is invertible.

5. $A = \begin{bmatrix} -9 & 3 \\ 6 & 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 7 & 4 \\ 14 & 8 \end{bmatrix}$

ANSWERS

CHECK YOUR ANSWERS

1. $\begin{bmatrix} 1 & 1 \\ 2 & \frac{5}{2} \end{bmatrix}$

2. $\begin{bmatrix} -\frac{1}{7} & \frac{2}{7} \\ -\frac{5}{7} & \frac{3}{7} \end{bmatrix}$

3. $\begin{bmatrix} 4 & 0 \\ -5 & -1 \end{bmatrix}$

4. The given matrices are multiplicative inverses of each other because the product of these matrices is the 3×3 identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. $|A| = -27 \neq 0$, so A is invertible.

6. $|A| = 0$, so A is noninvertible.

ARITHMETIC SEQUENCES

A **SEQUENCE** is an arrangement of numbers in a particular order.

A sequence can be **FINITE**, such as the sequence 1, 2, 3, 4, 5.

A sequence can be **INFINITE**, such as the sequence 1, 2, 3, . . .

The three dots (called an ellipsis) indicate the numbers keep going infinitely.

In both sequences above, the number 1 is the first term of the sequence, the number 2 is the second term of the sequence, and so on.

In general, the *n*th term of a sequence refers to the position of the term in the sequence. For example, the fifth term of both sequences above is 5.

An **ARITHMETIC SEQUENCE** is a sequence such that the difference d between consecutive terms is **constant**. The difference d is called the **COMMON DIFFERENCE** of the arithmetic sequence.

Both the finite sequence and the infinite sequence given on the previous page are arithmetic sequences with common difference $d = 1$.

Each term is 1 more than the previous term.

EXAMPLE: Determine if each of the following infinite sequences is arithmetic. Then write the next four terms of the sequence.

1. Sequence: 3, 7, 11, 15, 19, . . .

3 , 7 , 11 , 15 , 19 , . . .

+4 +4 +4 +4

common difference = 4

This is an arithmetic sequence because the difference between any pair of consecutive terms is always the same. In this case, the common difference is 4. The next four terms are 23, 27, 31, and 35.

2. Sequence: 20, 17, 14, 11, . . .

$20, 17, 14, 11, \ldots$

-3 -3 -3

common difference = -3

This is an arithmetic sequence because the difference between any pair of consecutive terms is always the same. In this case, the common difference is –3. The next four terms are 8, 5, 2, and –1.

3. Sequence: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

$-\frac{1}{4}$ $-\frac{1}{8}$ $-\frac{1}{16}$

no common difference

This is not an arithmetic sequence because there is no common difference.

Note: Even though this sequence is not arithmetic, there is a simple pattern. The next four terms of the sequence are $\frac{1}{32}, \frac{1}{64}, \frac{1}{128}$, and $\frac{1}{256}$. We will learn more about this type of sequence later in this unit.

To determine the value of any term in an arithmetic sequence, we can use the arithmetic sequence formula:

first term; common difference

$$a_n = a_1 + (n - 1)d.$$

nth term; position of previous term

In this formula, a_n is the ***nth term*** of the sequence.

Number of Term	1	2	3	4	5	. . .	a_n
Term of Sequence	a_1	$a_1 + d$	$a_1 + 2d$	$a_1 + 3d$	$a_1 + 4d$	. . .	$a_n = a_1 + (n - 1)d$

Let's look at the arithmetic sequence 1, 3, 5, 7, 9, 11, . . .

In this sequence, $a_1 = 1$ and the common difference is $d = 2$.

Therefore, $a_n = 1 + (n - 1)(2) = 1 + (2n - 2) = 2n - 1$.

The equation $a_n = 2n - 1$ is called the **EXPLICIT FORMULA** for the given sequence. We can use the explicit formula to find the value of any term in the sequence.

LINEAR EQUATIONS AND ARITHMETIC SEQUENCES

Questions about arithmetic sequences can easily be thought of as questions about lines and linear equations.

We can identify terms of the sequence with points on a line where the x-coordinate is the term number and the y-coordinate is the term itself.

Let's go back to the arithmetic sequence we looked at earlier:

1, 3, 5, 7, 9, 11, . . .

Recall that $a_1 = 1$, the common difference is $d = 2$, and $a_n = 2n - 1$.

In this sequence, the **first term** of the sequence is 1, so we can identify this term with the point (1, 1). The **second term** of the sequence is **3**, so we can identify it with the point (2, 3).

Let's list the points associated with this sequence:

(1, 1), (2, 3), (3, 5), (4, 7), (5, 9), (6, 11), . . .

Note that the common difference d is equal to the **slope** of the line that passes through any two of these points. For example, the slope of the line that runs from (1, 1) to (2, 3) is $\frac{3-1}{2-1} = 2 = d$.

Recall the POINT-SLOPE FORM of a linear equation:

$$y - y_1 = m(x - x_1).$$

slope

the coordinates of a point on the line

If we were to write an equation of the line passing through (1, 1) with slope 2 in point-slope form, we get the following:

$y - 1 = 2(x - 1)$ *Substitute (1, 1) for the point (x_1, y_1).*

$y - 1 = 2x - 2$ *Distributive Property*

$y = 2x - 1$ *Addition Property of Equality*

Compare $y = 2x - 1$ to the explicit formula $a_n = 2n - 1$.

They are the same except for the letters used to denote the unknowns.

EXAMPLE: Write an explicit formula for the *n*th term of the given infinite sequence. Then find the twentieth term of the sequence.

−50, −20, 10, 40, 70, 100, . . .

This sequence is arithmetic, with the first term, $a_1 = -50$, and the common difference, $d = -20 - (-50) = 30$.

−50 , −20 , 10 , 40 , 70 , 100 , . . .

+30 +30 +30 +30 +30

Step 1: Use the arithmetic sequence formula.

For the given sequence, the first term, $a_1 = -50$, and the common difference, $d = 30$. Substitute those values into the arithmetic sequence formula.

$a_n = a_1 + (n - 1)d$

$a_n = -50 + (n - 1) \cdot 30$

Step 2: Simplify.

$a_n = -50 + 30n - 30$ Distributive Property

$a_n = 30n - 80$ ← This is an explicit formula for the *n*th term of the sequence.

Step 3: To find the twentieth term, set $n = 20$ in the explicit formula for the given sequence.

$a_n = 30n - 80$

$a_{20} = 30 \cdot 20 - 80$

$a_{20} = 600 - 80$

$a_{20} = 520$

So, an explicit formula for this sequence is $a_n = 30n - 80$, and the twentieth term of the sequence is 520.

EXAMPLE: Given that the sixth term of an arithmetic sequence is 17 and the fourteenth term of the sequence is −7, find an explicit formula for the *n*th term of the sequence. Then use that formula to find the value of *n* for which the *n*th term of the sequence is −40.

There are two methods for finding the explicit formula.

Method 1: Identify the sequence with a Linear Equation.

Identify the two given terms with the points (6, 17) and (14, −7).

The slope of the line passing through these two points is $m = \frac{-7 - 17}{14 - 6} = \frac{-24}{8} = -3$.

Using the point (6, 17) and the slope $m = -3$, write an equation of a line in point-slope form:

$y - 17 = -3(x - 6)$

$y - 17 = -3x + 18$

$y = -3x + 35$

A formula for the corresponding arithmetic sequence is $a_n = -3n + 35$.

This is an explicit formula for the nth term of the given sequence.

Method 2: Use the arithmetic sequence formula.

Step 1: Substitute the given information into the arithmetic sequence formula to get a system of equations.

$a_n = a_1 + (n - 1)d$ *arithmetic sequence formula*

Equation 1:

$17 = a_1 + (6 - 1)d$ *The sixth term = 17, so $n = 6$ and $a_6 = 17$.*

$17 = a_1 + 5d$

Equation 2:

$a_{14} = a_1 + (14 - 1)d$ *The fourteenth term = -7, so $n = 14$ and $a_{14} = -7$.*

$-7 = a_1 + 13d$

System of Equations is: $17 = a_1 + 5d$
$-7 = a_1 + 13d$

Step 2: Solve the system of equations to find the first term of the sequence, a_1, and the common difference, d.

$24 = -8d$ *Subtract the second equation from the first.*

$-3 = d$

Find a_1 by substituting $d = -3$ into either equation.

$17 = a_1 + 5d$

$17 = a_1 + 5(-3)$ *Substitute $d = -3$*

$17 = a_1 - 15$

$32 = a_1$

Step 3: Write an explicit formula for the *n*th term of the sequence.

Substitute the first term, $a_1 = 32$, and the common difference, $d = -3$, into the arithmetic sequence formula.

$a_n = a_1 + (n - 1)d$ *arithmetic sequence formula*

$a_n = 32 + (n - 1)(-3)$

$a_n = 32 - 3n + 3$

$a_n = -3n + 35$

This is an explicit formula for the *n*th term of the given sequence.

Now use the explicit formula to find which term is equal to −40.

$a_n = -3n + 35$ *explicit formula*

$-40 = -3n + 35$ *Substitute* $a_n = -40$.

$-75 = -3n$

$25 = n$

So, $a_{25} = -40$, and therefore $n = 25$.

Therefore, an explicit formula for the *n*th term of the sequence is $a_n = -3n + 35$, and the 25th term of the sequence is −40: $n = 25$.

CHECK YOUR KNOWLEDGE

For questions 1 through 3, determine if each of the following infinite sequences is arithmetic. Then write the next four terms of the sequence.

1. 34, 64, 94, 124, . . .

2. −4, 3, 9, 14, 18, . . .

3. −9.2, −11.3, −13.4, −15.5, −17.6, . . .

For questions 4 and 5, find the first five terms and an explicit formula for the arithmetic sequence.

4. $a_1 = 31$, $d = 5$

5. $a_1 = \frac{1}{8}$, $d = -\frac{1}{2}$

For questions 6 and 7, write an explicit formula for the *n*th term of the given sequence. Then find the thirtieth term of the sequence.

6. −8, 1, 10, 19, . . .

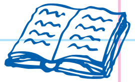
7. 3, 44, 85, 126, . . .

8. Given $a_4 = -13$ and $a_{11} = -27$, find an explicit formula for the *n*th term of the sequence. Then use that formula to find the value of *n* for which the *n*th term of the sequence is −47. Assume that the sequence is arithmetic.

9. Given that the seventh term of an arithmetic sequence is 11 and the twentieth term of the sequence is 30.5, find an explicit formula for the *n*th term of the sequence. Then use that formula to find the value of *n* for which the *n*th term of the sequence is 36.5.

ANSWERS

CHECK YOUR ANSWERS



1. This is an arithmetic sequence. The common difference is $d = 30$. The next four terms are 154, 184, 214, and 244.

2. This is not an arithmetic sequence. There is no common difference. The differences are 7, 6, 5, and 4. The next four terms are 21, 23, 24, and 24.

3. This is an arithmetic sequence. The common difference is $d = -2.1$. The next four terms are −19.7, −21.8, −23.9, and −26.

4. First five terms: 31, 36, 41, 46, 51, . . .
   Explicit formula: $a_n = 5n + 26$

5. First five terms: $\frac{1}{8}, -\frac{3}{8}, -\frac{7}{8}, -\frac{11}{8}, -\frac{15}{8}, \ldots$

   Explicit formula: $a_n = -\frac{1}{2}n + \frac{5}{8}$

6. $a_1 = -8$, $d = 9$
   Explicit formula: $a_n = 9n - 17$
   $a_{30} = 253$

7. $a_1 = 3$, $d = 41$
   Explicit formula: $a_n = 41n - 38$
   $a_{30} = 1{,}192$

8. $a_1 = -7$, $d = -2$
   Explicit formula: $a_n = -2n - 5$
   $a_{21} = -47$

9. $a_1 = 2$, $d = 1.5$
   Explicit formula: $a_n = 1.5n + 0.5$
   $a_{24} = 36.5$

ARITHMETIC SERIES

The sum of the terms of a sequence is called a **SERIES**.

A series is *arithmetic* if any two consecutive terms have the same difference. In other words, an **ARITHMETIC SERIES** is an expression formed by adding the terms of an arithmetic sequence.

There is a simple formula for a finite arithmetic series:

S_n = (# of terms) • (average of first and last term)

If the arithmetic sequence is $a_1, a_2, \ldots, a_n$, then we can write the arithmetic series as follows:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Since $a_n = a_1 + (n - 1)d$ (where d is the common difference of the arithmetic sequence), we can also write the formula as follows:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$= \frac{n}{2}(a_1 + a_1 + (n - 1)d)$$

$$= \frac{n}{2}(2a_1 + (n - 1)d)$$

Note: Either of the previous formulas can be used to find the sum of the terms of an arithmetic sequence.

EXAMPLE: Given the arithmetic sequence −5, 1, 7, 13, 19, . . . , find the sum of the first 30 terms.

Method 1: Use the formula $S_n = \frac{n}{2}(a_1 + a_n)$.

Step 1: Identify the first term, a_1, and the common difference, d. Then write an explicit formula for a_n.

$a_1 = -5$, $d = 6$

$a_n = a_1 + (n - 1)d$ *arithmetic sequence formula*

$a_n = -5 + (n - 1) \cdot 6$ Substitute $d = 6$ and $a_1 = -5$.

$a_n = 6n - 11$ ← This is an explicit formula for the nth term of the sequence.

Step 2: Find the thirtieth term of the arithmetic sequence using the explicit formula.

$a_n = 6n - 11$

$a_{30} = 6(30) - 11$

$a_{30} = 169$

Step 3: Find the sum of the first 30 terms using the arithmetic series formula.

$S_n = n\left(\frac{a_1 + a_n}{2}\right)$

$S_{30} = 30\left(\frac{-5 + 169}{2}\right)$ $a_1 = -5$ and $a_{30} = 169$

$S_{30} = 30\left(\frac{164}{2}\right) = 15(164) = 2{,}460$

$S_{30} = 2{,}460$ ← This is the sum of the first 30 terms of the arithmetic sequence.

Method 2: Use the formula $S_n = \frac{n}{2}(2a_1 + (n - 1)d)$.

$S_{30} = \frac{30}{2}(2(-5) + (30 - 1) \cdot 6)$ Substitute $n = 30$, $a_1 = -5$ and $d = 6$.

$S_{30} = 15(-10 + 29 \cdot 6)$

$S_{30} = 15(-10 + 174)$

$S_{30} = 15(164)$

$S_{30} = 2{,}460$

So, the sum of the first 30 terms of the arithmetic sequence is 2,460.

EXAMPLE: Given an arithmetic sequence with $a_{10} = -48$ and $a_{19} = -84$, find the sum of the first 25 terms.

Method 1: Identify the sequence with a linear equation.

Step 1: Identify the two given terms with the points (10, −48) and (19,−84). Then write an equation for the line that passes through them in point-slope form.

The slope of the line passing through these two points is $m = \frac{-84 - (-48)}{19 - 10} = \frac{-36}{9} = -4$.

Using the point (10, −48) and the slope $m = -4$, we can write an equation of a line in point-slope form:

$y - (-48) = -4(x - 10)$
$y + 48 = -4x + 40$
$y = -4x - 8$

Step 2: Write the corresponding arithmetic sequence formula.

Given the linear equation of $y = -4x - 8$, a formula for the corresponding arithmetic sequence is $a_n = -4n - 8$.

This is an explicit formula for the nth term of the sequence.

Step 3: Find the twenty-fifth term of the arithmetic sequence using the explicit formula.

$a_n = -4n - 8$
$a_{25} = -4(25) - 8$
$a_{25} = -108$

Step 4: Calculate the sum of the first 25 terms using the arithmetic series formula.

$$S_n = n\left(\frac{a_1 + a_n}{2}\right)$$

$$S_{25} = 25\left(\frac{-12 - 108}{2}\right)$$

$$S_{25} = 25\left(\frac{-120}{2}\right) = 25(-60) = -1{,}500$$

$S_{25} = -1{,}500$ ← *This is the sum of the first 25 terms of the arithmetic sequence.*

So, the sum of the first 25 terms of this arithmetic sequence is −1,500.

Method 2: Use the arithmetic sequence formula.

Step 1: Find an explicit formula for the *n*th term of this sequence by writing a system of equations and solving for the common difference and the first term.

$a_n = a_1 + (n - 1)d$ *arithmetic sequence formula*

System of Equations

Equation 1: $-48 = a_1 + 9d$ $a_{10} = -48, n = 10$
Equation 2: $-84 = a_1 + 18d$ $a_{19} = -84, n = 19$

Step 2: Solve the system of equations to find the first term of the sequence, a_1, and the common difference, d.

$$\begin{array}{r} -48 = a_1 + 9d \\ -\ (-84 = a_1 + 18d) \\ \hline \frac{36}{-9} = \frac{-9d}{-9} \end{array}$$

Subtract the second equation from the first.

$-4 = d$

Find a_1 by substituting $d = -4$ into either equation.

$-48 = a_1 + 9d$

$-48 = a_1 + 9(-4)$ *Substitute* $d = -4$.

$-48 = a_1 - 36$

$-12 = a_1$

Step 3: Write an explicit formula for the *n*th term of the sequence.

Substitute the first term, $a_1 = -12$, and the common difference, $d = -4$, into the arithmetic sequence formula.

$a_n = a_1 + (n - 1)d$ *arithmetic sequence formula*

$a_n = -12 + (n - 1)(-4)$

$a_n = -12 - 4n + 4$

$a_n = -4n - 8$ ← *This is an explicit formula for the nth term of the sequence.*

Step 4: Find the twenty-fifth term of the arithmetic sequence using the explicit formula.

$a_n = -4n - 8$

$a_{25} = -4(25) - 8$

$a_{25} = -108$

Step 5: Find the sum of the first 25 terms using the arithmetic series formula.

$S_n = n\left(\frac{a_1 + a_n}{2}\right)$

$S_{25} = 25\left(\frac{-12 - 108}{2}\right)$

$S_{25} = 25\left(\frac{-120}{2}\right) = 25(-60) = -1{,}500$

$S_{25} = -1{,}500$ ← *This is the sum of the first 25 terms of the arithmetic sequence.*

So, the sum of the first 25 terms of this arithmetic sequence is −1,500.

EXAMPLE: A new outdoor theater at Breaker Park has 100 rows of seating. The first row has 14 seats, and then each row after the first row has 2 more seats than the row in front of it. What is the seating capacity at the new Breaker Park outdoor theater?

To find the seating capacity at this outdoor theater, we can use the arithmetic series formula.

Think: The arithmetic sequence that represents the number of seats in each row of the theater is

14, 16, 18, 20, 22, 24, ...
+2 +2 +2 +2 +2

$a_1 = 14$ and $d = 2$

Step 1: Find the explicit formula for the given sequence.

$a_n = a_1 + (n - 1)d$ arithmetic sequence formula

$a_n = 14 + (n - 1) \cdot 2$

$a_n = 2n + 12$ ← This is an explicit formula for the nth term of the sequence.

Step 2: Find the one-hundredth term of the arithmetic sequence using the explicit formula.

$a_n = 2n + 12$

$a_{100} = 2(100) + 12$

$a_{100} = 212$

Step 3: Find the sum of the 100 terms using the arithmetic series formula.

$$S_n = n\left(\frac{a_1 + a_n}{2}\right)$$

$$S_{100} = 100\left(\frac{14 + 212}{2}\right)$$

$$S_{100} = 100\left(\frac{226}{2}\right) = 100(113) = 11{,}300$$

← This is the sum of the first 100 terms of the arithmetic sequence.

So, the seating capacity at the new Breaker Park outdoor theater is 11,300 seats.

CHECK YOUR KNOWLEDGE

1. Find the sum of the first 36 terms of the arithmetic sequence with the first term, $a_1 = 15$, and the common difference, $d = 2.5$.

2. Find the sum of the first 50 positive integers.

3. Given the arithmetic sequence −12, −7, −2, 3, 8, . . . , find the sum of the first 40 terms.

4. Given the arithmetic sequence with $a_8 = -45$ and $a_{17} = -90$, find the sum of the first 30 terms.

5. Given the arithmetic sequence with $a_{11} = 58$ and $a_{16} = 78$, find the sum of the first 76 terms.

6. An Olympic team is raising money for a local community food bank. They place $0.25 in a jar the first day. Each day after the first, they place $0.05 more than the previous day for a total of 365 days. How much money will the team have to donate to the food bank after the entire year?

7. What is the difference between an arithmetic sequence and an arithmetic series?

ANSWERS

CHECK YOUR ANSWERS

1. $S_{36} = 2{,}115$

2. $S_{50} = 1{,}275$

3. $S_{40} = 3{,}420$

4. $S_{30} = -2{,}475$

5. $S_{76} = 12{,}768$

6. Arithmetic sequence: 0.25, 0.30, 0.35, 0.45, 0.55, 0.60, . . .
   $a_1 = 0.25$, $d = 0.05$, $a_{365} = 18.45$
   $S_{365} = 3{,}413$

   So, the Olympic team will be able to donate $3,413 to their community food bank.

7. Responses may vary. An arithmetic sequence is an ordered list of numbers in which the difference between consecutive terms is constant. An arithmetic series is a sum of the terms of an arithmetic sequence.

GEOMETRIC SEQUENCES

A **GEOMETRIC SEQUENCE** is a sequence of numbers such that the quotient r between consecutive terms is constant. The number r is called the **COMMON RATIO** of the geometric sequence.

Here is an example of a geometric sequence with common ratio $r = 2$.

1 , 2 , 4 , 8 , 16 , 32 , . . .

×2 ×2 ×2 ×2 ×2

geometric sequence

To determine the value of any term in a geometric sequence, we can use the geometric sequence formula:

$$a_n = a_1 r^{n-1}.$$

In this formula, a_n is the ***n*th term** of the sequence.

Number of Term	1	2	3	4	5	...	a_n
Term of Sequence	a_1	$a_1 r$	$a_1 r^2$	$a_1 r^3$	$a_1 r^4$	...	$a_1 r^{n-1}$

For example, in the geometric sequence 1, 2, 4, 8, 16, 32, . . . , $a_1 = 1$ and $r = 2$.

We can substitute these values into the geometric sequence formula to find an **explicit formula** for the sequence.

$a_n = a_1 r^{n-1}$ geometric sequence formula

$a_n = 1 \cdot 2^{n-1}$

$a_n = 2^{n-1}$

So, $a_n = 2^{n-1}$ is an explicit formula for this geometric sequence.

EXAMPLE: The first term of a sequence is 3, and each term after the first is three times the preceding term. What is the *n*th term of the sequence?

3, 9, 27, . . .

The first term and the common ratio of the sequence are both 3.

$a_1 = 3$ and $r = 3$

Use the geometric sequence formula to find the *n*th term of the sequence.

$a_n = a_1 r^{n-1}$

$a_n = 3 \cdot 3^{n-1}$

This is an explicit formula for the nth term of the geometric sequence.

So, the *n*th term of the geometric sequence is given by the formula $a_n = 3 \cdot 3^{n-1}$.

EXAMPLE: Write an explicit formula for the *n*th term of the following geometric sequence. Then find the ninth term of the sequence.

2, −6, 18, −54, . . .

Step 1: Identify the common ratio, *r*, and the first term, a_1.

Common ratio: $r = \frac{-6}{2} = -3$

First term: $a_1 = 2$

Step 2: Write an explicit formula for this geometric sequence by substituting the common ratio and the first term into the geometric sequence formula.

$a_n = a_1 r^{n-1}$

$a_n = 2(-3)^{n-1}$

Step 3: Find the ninth term by substituting $n = 9$ into the explicit formula for the given sequence.

$a_n = 2(-3)^{n-1}$

$a_9 = 2(-3)^{9-1}$

$a_9 = 2(-3)^8$

$a_9 = 2 \cdot 6{,}561$ *Use a calculator to find $(-3)^8 = 6{,}561$.*

$a_9 = 13{,}122$

So, an explicit formula for this geometric sequence is $a_n = 2(-3)^{n-1}$, and the ninth term of the sequence is 13,122.

EXAMPLE: Harper purchases a car for $28,600. An online car appraisal states that her new car will depreciate at a rate of 11% every year. If that assumption does not change over the course of Harper's ownership, to the nearest dollar, how much will her car be worth in year five?

To solve this problem, use a geometric sequence and the geometric sequence formula.

Step 1: Identify the geometric sequence. Then identify the common ratio, r, and the first term, a_1.

> **Think:** Since the car depreciates at 11% per year, that means its value will be 89% of the prior year's value: 100% – 11% = 89%.

Geometric sequence: \$28,600, \$25,454, \$22,654.06, . . .

Common ratio: $r = 0.89$

First term: $a_1 = 28{,}600$

Step 2: Substitute the common ratio and the first term into the geometric sequence formula.

$a_n = a_1 r^{n-1}$

$a_n = 28{,}600 \cdot 0.89^{n-1}$

This is an explicit formula for the nth term of the geometric sequence.

Step 3: Find the fifth term of the sequence (corresponding to the value of the car in the fifth year) by substituting $n = 5$ into the explicit formula found in Step 2.

$a_n = 28{,}600 \cdot 0.89^{n-1}$

$a_5 = 28{,}600 \cdot 0.89^{5-1}$

$a_5 = 28{,}600 \cdot 0.89^4$

$a_5 = 28{,}600 \cdot 0.62742241$

$a_n \approx 17{,}944$

So, to the nearest dollar, Harper's car will be worth \$17,944 in five years.

FINDING THE COMMON RATIO USING NONCONSECUTIVE TERMS

If we know two consecutive terms of a geometric sequence, we can find the common ratio simply by dividing the second known term by the first known term.

However, what if we know two terms of a geometric sequence, but they are *not* consecutive?

In this case, we can still find the common ratio quickly, as follows.

Suppose a_n and a_m are the *n*th and *m*th term of a geometric sequence with positive terms and $m > n$. Then the common ratio of the geometric sequence is

$$r = \sqrt[m-n]{\frac{a_m}{a_n}} = \left(\frac{a_m}{a_n}\right)^{\frac{1}{m-n}}$$

In words, this formula says, "Divide the second known term by the first known term, and then take a root."

Which root do we take? Take the distance between the two term numbers, and that's the root we use.

Let's go back to the sequence 1, 2, 4, 8, 16, 32, . . .

Note that the second term of this sequence is 2 and the fifth term of this sequence is 16. Let's find the common ratio of the sequence using the second and fifth terms.

Step 1: Divide the fifth term by the second term: $16 \div 2 = 8$.

Step 2: Find the distance between the two term numbers: $5 - 2 = 3$.

Step 3: Take the appropriate root (in this case the cube root): $\sqrt[3]{8} = 2$.

So, the common ratio is 2.

Note that if we substitute the numbers into the formula directly, we get:

$$r = \sqrt[5-2]{\frac{16}{2}} = \sqrt[3]{8} = 2.$$

EXAMPLE: What is the third term of the geometric sequence whose second term is $\frac{1}{9}$ and whose fourth term is $\frac{1}{81}$?

Use the formula $r = \sqrt[m-n]{\frac{a_m}{a_n}}$.

We are given that $a_2 = \frac{1}{9}$ and $a_4 = \frac{1}{81}$ (so that $m = 4$ and $n = 2$).

$$r = \sqrt[4-2]{\frac{1}{81} \div \frac{1}{9}}$$

$$r = \sqrt{\frac{1}{81} \cdot \frac{9}{1}}$$

$$r = \sqrt{\frac{1}{9}}$$

$$r = \frac{1}{3}$$

Since the second term is $\frac{1}{9}$ and the common ratio is $\frac{1}{3}$, the third term is $a_3 = \left(\frac{1}{9}\right)\left(\frac{1}{3}\right) = \frac{1}{27}$.

CHECK YOUR KNOWLEDGE

For questions 1 through 3, find an explicit formula for the *n*th term of the geometric sequence. Then find the eighth term of the sequence.

1. 2, 6, 9, 27, . . .

2. −10, 25, −62.5, 156.25, . . .

3. $\frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$

4. What is the fourth term of the geometric sequence with only positive terms whose second term is 45 and whose sixth term is 3,645?

5. What is the sixth term of the geometric sequence with only negative terms whose second term is −5 and whose fourth term is $-\frac{1}{5}$? (Hint: Since all the terms are negative, the procedure is the same as if all the terms were positive.)

6. The population of Berryville is 176,082. The census establishes that the town's population will increase at a rate of 2% every year for the next 10 years. If that assumption does not change over 10 years, what is the predicted population of Berryville in 10 years? Round your answer to the nearest whole number.

ANSWERS

CHECK YOUR ANSWERS

1. $a_n = 2 \cdot 3^{n-1}$

$a_8 = 4{,}374$

2. $a_n = -10 \cdot (-2.5)^{n-1}$

$a_8 = 6{,}103.515625$

3. $a_n = \frac{1}{8}\left(\frac{1}{2}\right)^{n-1}$

$a_8 = \frac{1}{1{,}024}$

4. $a_4 = 405$

5. $a_6 = -\frac{1}{125}$

6. The population of Berryville in 10 years to the nearest whole number will be approximately 210,434 people.

GEOMETRIC SERIES

A **GEOMETRIC SERIES** is the sum of the terms of a geometric sequence.

If a is the first term of the geometric sequence, and r is the common ratio, and $r \neq 1$, then the sum of the first n terms of the geometric sequence is

the geometric series formula → $S_n = \frac{a(1-r^n)}{1-r}$.

EXAMPLE: Given the geometric sequence 3, −6, 12, −24, . . . , find the sum of the first six terms.

Step 1: Identify the first term and the common ratio.

First term: $a = 3$

Common ratio: $r = -6 \div 3 = -2$

Step 2: Use the geometric series formula to find the sum of the first six terms of the given geometric sequence.

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_6 = \frac{3(1-(-2)^6)}{1-(-2)}$$

$$S_6 = \frac{3(1-64)}{3}$$

$$S_6 = \frac{\cancel{3}(-63)}{\cancel{3}}$$

$$S_6 = -63$$

So, the sum of the first six terms of the given geometric sequence is −63.

EXAMPLE: The common ratio of a geometric sequence is −5 and the sum of the first six terms is 20,832. What is the value of the first term?

We can use the geometric series formula to find the first term in this geometric sequence.

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$20{,}832 = \frac{a(1-(-5)^6)}{1-(-5)}$$ Substitute $S_6 = 20{,}832$ and $r = -5$.

$$20{,}832 = \frac{a(1-15{,}625)}{1+5}$$

$$20{,}832 = \frac{a(-15{,}624)}{6}$$

$$\frac{6(20{,}832)}{-15{,}624} = a$$

$$\frac{124{,}992}{-15{,}624} = a$$

$$-8 = a$$

So, the first term of the geometric sequence is $a = -8$.

EXAMPLE: Carla's Boutique made a profit of $3,000 in January on the sale of graphic T-shirts. Its profits each month are 4% more than the profits of the prior month for the rest of the year. To the nearest dollar, what is the total profit Carla's Boutique generates from the sale of graphic T-shirts throughout this year?

To solve this problem, use the geometric series formula.

Step 1: Identify the first term, a, and the common ratio, r.

First term: $a = 3{,}000$

Common ratio: $r = 1.04$

Think: The profits in each month are 4% greater than the profits from the prior month: 100% + 4% = 104%. As a decimal, this is 1.04.

The geometric sequence is

3,000, 3,120, 3,244.80, 3,374.592, ...

We find each term by multiplying the prior term by 1.04.

Step 2: Use the geometric series formula to find the sum of the first 12 terms (corresponding to the 12 months of the year) of the given geometric sequence.

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_{12} = \frac{3{,}000(1-(1.04)^{12})}{1-(1.04)}$$

$$S_{12} \approx 45{,}077.41639$$

So, the total profit Carla's Boutique generates from the sale of graphic T-shirts throughout this year to the nearest dollar is $45,077.

So far we have learned how to find the sum of a geometric series with a last term. We can also find the sum of infinite series (which have no last term).

The sum S of an **infinite geometric series** with the first term, a, and the common ratio, r, where $-1 < r < 1$, is

$$S = \frac{a}{1-r}.$$

If $r \geq 1$ or $r \leq -1$, then the sum does not exist.

EXAMPLE: The first term of an infinite geometric series is –10 and the common ratio is 0.5. What is the sum of this infinite geometric series?

Since the common ratio is between –1 and 1, we can use the formula $S = \frac{a}{1-r}$.

First term: $a = -10$
Common ratio: $r = 0.5$

$$S = \frac{a}{1-r}$$

$$S = \frac{-10}{1-0.5}$$

$$S = \frac{-10}{0.5}$$

$$S = -20$$

So, the sum of this infinite geometric series is –20.

EXAMPLE: The first term of an infinite geometric series is 9 and the sum is 45. What is the common ratio of this infinite geometric series?

Use the formula $S = \frac{a}{1-r}$.

First term: $a = 9$
Sum: $S = 45$

$$S = \frac{a}{1-r}$$

$$45 = \frac{9}{1-r}$$

$$45(1 - r) = 9$$

$$45 - 45r = 9$$

$$-45r = -36$$

$r = \frac{-36}{-45}$

$r = \frac{4}{5}$

So, the common ratio of this infinite geometric series is $\frac{4}{5}$.

CHECK YOUR KNOWLEDGE

For questions 1 and 2, find the sum of the first eight terms of the given geometric sequence.

1. −4, 8, −16, 32, −64, 128 . . .

2. 24, 12, 6, 3, 1.5, 0.75 . . .

3. The common ratio of a geometric sequence is −4 and the sum of the first seven terms is 19,662. What is the value of the first term of the sequence?

4. One year, a popular musician donates 15,000 books to community libraries. The musician commits to donating 10% more books than the prior year for the next 7 years. What is the total number of books the musician will have donated after these 8 years? Round your answer to the nearest hundred books.

5. The first term of an infinite geometric series is −12 and the common ratio is 0.4. What is the sum of this infinite geometric series?

6. The first term of an infinite geometric series is 24 and the sum is −6. What is the common ratio of this infinite geometric series?

ANSWERS

CHECK YOUR ANSWERS

1. $S_8 = 340$

2. $S_8 = 47.8125$

3. $a = 6$

4. The musician will donate approximately 171,500 books.

5. $S = -20$

6. $r = 5$

PERMUTATIONS AND COMBINATIONS

The **COUNTING PRINCIPLE** states that if there are a ways to do one thing, and b ways to do another thing, then there are ab ways to do *both* things.

For example, if a sweater comes in 3 colors and 4 sizes, then there are $3 \cdot 4 = 12$ possible combinations of color and size for the sweater.

PERMUTATIONS VERSUS COMBINATIONS

A **PERMUTATION** is an arrangement of objects in a specific order. For example, HTMA is a permutation of the letters in the word MATH.

Permutations are used when the **order is important** (different order = different object).

A **COMBINATION** is a choice of objects from a collection of objects. For example, a council of 3 people from a group of 10 is a combination.

Combinations are used when the order is NOT important (different order = same object).

PERMUTATIONS WITH REPETITION

Imagine we have a bag of n distinct objects. We want to arrange them in a line, allowing the same object to be used more than once.

For example, if we start with $n = 4$ objects, we have 4 choices each time.

So, if we choose 5 times, then the number of possible arrangements, with repetition allowed, is $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$.

In general, the number of ways to arrange r objects from a choice of n objects, with repetition allowed, is

$$n^r = \underbrace{n \cdot n \cdot \ldots \cdot n}_{r \text{ times}}$$

For example, the number of ways to create a 3-digit number, where each digit can be chosen from the numbers 1, 2, 3, 4, and 5, is $5 \cdot 5 \cdot 5 = 5^3$.

> **Think:** Since selections can be repeated, the number of choices stays the same each time.
>
> For example, 112, 121, and 211 are different permutations, with repetition allowed, of the numbers 1 and 2.

EXAMPLE: Paco can create a 6-digit code as the password for his new tablet by choosing each digit from the 10 possible digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. He can use each digit as many times as he would like. How many such passcodes can Paco choose from?

Since the order matters and Paco can repeat the digits, we count the number of permutations, with repetition:

$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^6$

$10^6 = 1{,}000{,}000$ ← WOW, that is a lot of possibilities!

> **Think:** Paco can choose from 10 digits for each of the 6 digits of his password.

So, there are 1,000,000 possible passwords for Paco to choose from.

PERMUTATIONS WITHOUT REPETITION

When there are *n* objects, and we wish to arrange them but without ever repeating an object, then we have *n* choices the first time, but we *must reduce the number of available choices by* 1 each time.

For example, in how many ways can 7 colored pencils, each a different color, be arranged?

> **Think:** The number of choices is reduced by 1 each time. Therefore, there are 7 choices for the first pencil, 6 choices for the second pencil, 5 choices for the third pencil, and so on.

Since the order matters but none of the pencils can repeat in our arrangement, we count the number of permutations, *without* repetition:

$7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040$

So, there are 5,040 permutations (or arrangements).

We can write this mathematically using **FACTORIAL FUNCTIONS**.

> The factorial function (!) means to multiply all positive integers from a given integer down to 1.
>
> This is read as "five factorial." → 5! means $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
>
> Note: 0! = 1

$n!$ (read as "n factorial") is used to compute the number of ways to arrange n objects. There is NO repetition of choices, AND the order matters. In this case, we are counting the number of ways to arrange ALL the objects.

EXAMPLE: Coach Bart has 9 players for the team's opening-game starting lineup. How many different batting orders can Coach Bart choose from for the opening game?

$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362{,}880$

So, Coach Bart has 362,880 different batting orders to choose from for the opening game.

We may want to arrange **fewer than the total number of objects**. When order matters, we can use the following formula:

number of objects to choose from → $n!$

$${}_nP_r = \frac{n!}{(n-r)!}$$

← number of objects chosen: $(n-r)!$

We can also use the notation $P(n, r)$.

$P(7, 4) = {}_7P_4$ and represents the number of permutations of 7 objects taken 4 at a time.

So, if we wanted to select only 4 of the 7 colored pencils from the earlier example, we could write the number of permutations as follows:

$${}_7P_4 = \frac{7!}{(7-4)!} = \frac{7!}{3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 840$$

This means the number of permutations of n (7) things taken r (4) at a time.

So, there are 840 permutations.

Think: We can also compute ${}_7P_4$ by starting with 7, then multiplying by 6, then 5, and then 4 to get $7 \cdot 6 \cdot 5 \cdot 4 = 840$. The first number in ${}_7P_4$ tells us to start with the number 7 and the second number in ${}_7P_4$ tells us to write down 4 factors.

As another example, ${}_{11}P_3 = 11 \cdot 10 \cdot 9 = 990$.

EXAMPLE: Joanna has 15 pairs of cartoon socks and wants to wear a different pair each day for 7 consecutive days. How many different arrangements of the 15 pairs of socks can Joanna choose from?

We can use the formula ${}_nP_r = \frac{n!}{(n-r)!}$ to answer this question.

${}_{15}P_7$ or $P(15, 7)$ means 15 objects taken 7 at a time.

$$_{15}P_7 = \frac{15!}{(15-7)!} = \frac{15!}{8!}$$

$$= \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot \cancel{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}$$

$$= 32{,}432{,}400$$

So, there are 32,432,400 different arrangements of the 15 pairs of socks Joanna can choose from.

COMBINATIONS WITHOUT REPETITION

Recall that a combination is a choice of objects from a collection of objects. When making such a choice, the order does *not* matter (different order = same object).

The combination formula can be written in terms of the permutation formula:

$$_nC_r = \frac{_nP_r}{r!}$$

We can also use $C(n, r)$.

Think: Since $_nP_r = \frac{n!}{(n-r)!}$, we have

$$_nC_r = \frac{_nP_r}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{(n-r)!} \cdot \frac{1}{r!} = \frac{n!}{r!\,(n-r)!}.$$

For example, in how many ways can you choose 6 numbers from a set of 10 numbers?

This is a combination because the set 1, 2, 3, 4, 5, 6 is the same as the set 6, 5, 4, 3, 2, 1 or any other arrangement of those 6 numbers.

Use the combination formula:

$$_nC_r = \frac{_nP_r}{r!} = \frac{_{10}P_6}{6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{6 \cdot 5}}{\cancel{6 \cdot 5} \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 210$$

EXAMPLE: Every Friday night, the Kelly family has dinner at their favorite restaurant. On Fridays, the chef offers all families a free "Build Your Own Family-Style Salad" with a menu of 8 different options. Each family must choose 5 of the 8 options. In how many different ways can the Kelly family build their salad?

Since the order of options isn't important, this is a combination.

$$_8C_5 = \frac{_8P_5}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5 \cdot 4}}{\cancel{5 \cdot 4} \cdot 3 \cdot 2 \cdot 1} = \frac{336}{6} = 56$$

So, the Kelly family can build their salad in 56 different ways.

CHECK YOUR KNOWLEDGE

1. In how many ways can you arrange 6 different perennial plants in a garden?

2. The Banker family installs a passcode lock on the entrance to their home. The passcode needs to be 5 digits, with no repeated digits. The digits the Bankers can choose from are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. How many such codes can they choose from?

3. How many different 4-letter words can you make by arranging 4 of the letters in the word COMPUTER?

4. Twenty students are running for student council. In how many ways can a president, vice president, secretary, and treasurer be chosen from the 20 students?

5. The student council is putting together a committee of 7 students from a class of 18 students to organize the annual end-of-year field day. In how many ways can the student council select the students from the class?

6. In the spring semester, 24 students sign up for the chess club. How many different ways are there to form a pair of 2 students from the 24?

7. Gilda has 15 different jackets and wants to donate 4 of them to a clothing drive. How many ways can Gilda choose 4 of her 15 jackets to donate?

8. Explain the difference between a permutation and a combination.

ANSWERS

CHECK YOUR ANSWERS

1. ${}_{6}P_{6} = 6! = 720$

2. ${}_{10}P_{5} = \frac{10!}{5!} = 30{,}240$

3. ${}_{8}P_{4} = \frac{8!}{4!} = 1{,}680$

4. ${}_{20}P_{4} = \frac{20!}{16!} = 116{,}280$

5. ${}_{18}C_{7} = \frac{{}_{18}P_{7}}{7!} = 31{,}824$

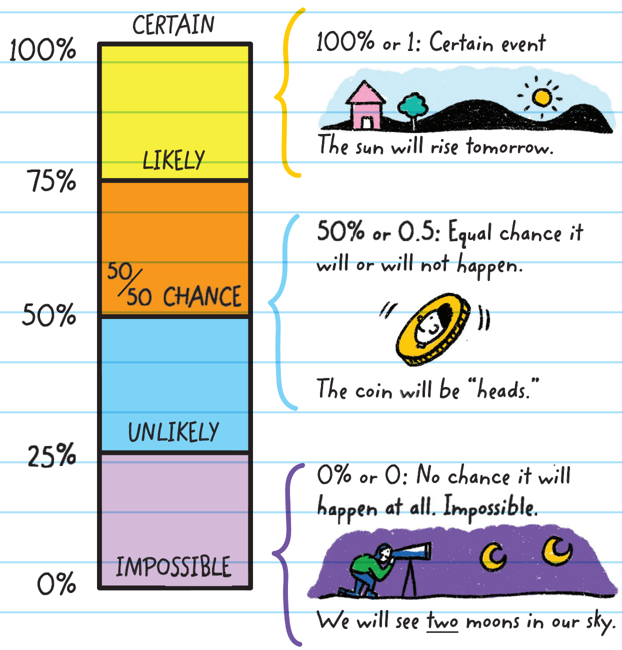
6. ${}_{24}C_{2} = \frac{{}_{24}P_{2}}{2!} = 276$

7. ${}_{15}C_{4} = \frac{{}_{15}P_{4}}{4!} = 1{,}365$

8. Responses may vary. In a permutation, the order of the objects is important. In a combination, the order is not important.

BASIC PROBABILITY

PROBABILITY is the likelihood that something will happen. More formally, the **PROBABILITY OF AN EVENT** is a number between 0 and 1 expressing how likely that event is to occur. If the probability of the event is 0, then it is **IMPOSSIBLE** for the event to occur. If the probability of the event is 1, then it is **CERTAIN** that the event will occur.



Probability can be expressed as a fraction, a decimal, or a percentage.

When we flip two quarters, each one can land on heads or on tails.

The **ACTION** (or **EXPERIMENT**) is what is happening. → flipping 2 quarters

The **OUTCOMES** are ALL the possible results. → both heads, one heads and one tails, or both tails

An **EVENT** is any outcome or group of outcomes. → for example, "both quarters land on heads" {HH}

We sometimes refer to the particular outcome whose probability we are evaluating as a **FAVORABLE OUTCOME**.

Consider the ACTION of flipping a single quarter. In this case, there are 2 OUTCOMES: "the quarter lands on heads" {H} and "the quarter lands on tails" {T}. Both of these outcomes are equally likely to occur.

When trying to find the probability P of an event where all outcomes are equally likely, we can use the **PROBABILITY FORMULA**.

$$P(\text{event}) = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}}$$

When flipping a single quarter, what is the probability that it lands on heads?

$$P(H) = \frac{\text{number of favorable outcomes } \{H\}}{\text{number of possible outcomes } \{H, T\}} = \frac{1}{2} = 0.5 = 50\%$$

So, there is a 50% chance the quarter will land on heads.

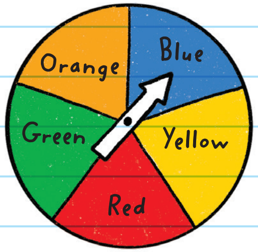
EXAMPLE: What is the probability of Cary choosing a blueberry muffin from a selection of 4 types of muffins (blueberry, bran, cranberry, oatmeal)? Assume that Cary is equally likely to choose each type of muffin.

The number of favorable outcomes (choosing a blueberry muffin) is 1, and the number of all possible outcomes (blueberry, bran, cranberry, oatmeal) is 4.

$$P(\text{blueberry}) = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}} = \frac{1}{4} = 0.25 = 25\%$$

So, there is a 25% chance that Cary will choose a blueberry muffin.

EXAMPLE: What is the probability of the spinner landing on blue or red?



In this example, we are considering the event E = {blue, red}.

$$P(\{\text{blue, red}\}) = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}} = \frac{2}{5} = 0.4 = 40\%$$

So, there is a 40% chance that the spinner will land on blue or red.

The **SAMPLE SPACE** for an experiment is the collection of all possible outcomes in that experiment.

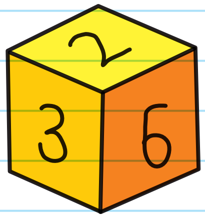
For example, consider the experiment of flipping a coin twice. The sample space (the collection of all possible outcomes) can be organized as a list.

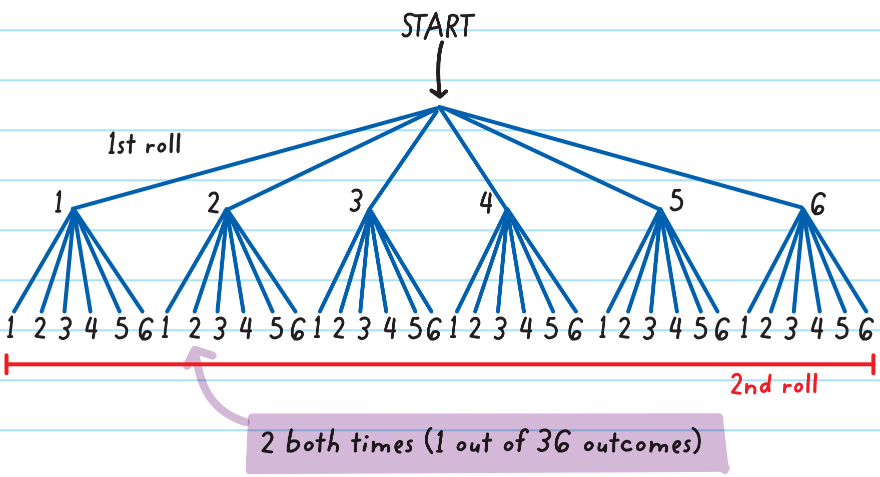
Outcome of the 1st flip	Outcome of the 2nd flip	Combination of the 2 flips
heads	heads	heads, heads
heads	tails	heads, tails
tails	heads	tails, heads
tails	tails	tails, tails

A **TREE DIAGRAM** is a type of visual representation that shows all possible outcomes of one or more events.

EXAMPLE: Gia rolls a 6-sided number cube twice. What is the probability that she rolls a 2 both times?

Record all possible outcomes in a tree diagram.





Then use the probability formula:

$P(\text{event}) = \dfrac{\text{number of favorable outcomes}}{\text{number of possible outcomes}}$

$P(\text{rolling a 2 both times}) = \dfrac{1}{36}$

COMPLEMENT OF AN EVENT

The **COMPLEMENT OF AN EVENT** is another event consisting of all outcomes in the sample space that are NOT in the given event. Informally, we can think of the complement of an event as the "opposite" event.

EVENT	COMPLEMENT
Heads	Tails
Win	Lose
Rain	No rain

Probability of an event + probability of its complement = 1

OR

Probability of an event + probability of its complement = 100%

In other words, there is a 100% chance that either an event or its complement will happen.

Let A be an event and let $\bar{A}$ be the complement of event A.

Rule of Complementary Events:

$$P(A) + P(\bar{A}) = 1$$
$$P(\bar{A}) = 1 - P(A)$$
$$P(A) = 1 - P(\bar{A})$$

For example, consider the sample space of all students in a class.

Let A be the event consisting of students in the class who wear glasses.

Then $\bar{A}$ (the complement of A) consists of students in the class who do not wear glasses.

Assume that the probability that a student chosen at random from the class wears glasses is 62% (in other words, $P(A) = 0.62$). Then the probability that a student chosen at random from the class does not wear glasses is 38% ($P(\bar{A}) = 0.38$).

62% + P(not wearing glasses) = 100%

P(not wearing glasses) = 38%

COMPOUND EVENTS

A **COMPOUND EVENT** is an event that consists of two or more single events occurring in succession.

A compound event can consist of INDEPENDENT EVENTS or DEPENDENT EVENTS.

Independent Events

Two events are **independent** if the occurrence of one of the events has no effect on whether the other event occurs.

If two events that make up a compound event are independent, we can compute the probability of the compound event by **multiplying** the probabilities of the two independent events.

If A and B are independent events, then

$P(A \text{ and } B) = P(A) \cdot P(B)$.

Equivalently,

$P(A \cap B) = P(A) \cdot P(B)$.

Remember: The symbol $\cap$ means intersection.

EXAMPLE: A bag of marbles contains 6 red marbles, 5 green marbles, 9 blue marbles, and 7 yellow marbles. A marble is drawn, and then it is put back into the bag and a second marble is drawn. What is the probability of drawing a red marble followed by a blue marble? Express the answer as a fraction.

Since this situation is a compound event consisting of two independent events, we can compute the desired probability by multiplying the probabilities of the two independent events.

Probability of each event:

$$P(\text{red}) = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}} = \frac{6}{27} = \frac{2}{9}$$

$$P(\text{blue}) = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}} = \frac{9}{27} = \frac{3}{9}$$

Probability of compound event:

$$P(\text{red, blue}) = P(A) \cdot P(B) = \frac{2}{9} \cdot \frac{3}{9} = \frac{6}{81} = \frac{2}{27}$$

So, the probability of drawing a red marble followed by a blue marble is $\frac{2}{27}$.

Dependent Events

Two events are **dependent** if the first event affects the probability of the second event.

If the events are dependent, multiply the probability of the first event by the probability of the second event given that the first event has happened.

If A and B are dependent events, then

$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$.

Equivalently,

$P(A \cap B) = P(A) \cdot P(B \text{ given } A)$.

EXAMPLE: Carmen has 5 cards, each printed with one of the vowels A, E, I, O, and U. Her friend selects one, doesn't replace it, and then selects again. What is the probability of selecting the A card, followed by the O card? Express your answer as a percent.

Event 1: Selecting the A card.
Event 2: Selecting the O card after an A card is removed.

Think: Since the first card is not being replaced, event 2 DEPENDS on event 1.

Probability of each event:

$$P(A) = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}} = \frac{1}{5}$$

$$P(O \text{ given } A) = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes remaining}} = \frac{1}{4}$$

(there is 1 less card)

Probability of compound event:

$$P(A \cap O) = P(A) \cdot P(O \text{ given } A) = \frac{1}{5} \cdot \frac{1}{4} = \frac{1}{20} = 0.05 = 5\%$$

So, the probability of selecting the A card followed by the O card is 5%.

ADDITION PROBABILITY RULE

If events A and B are mutually exclusive (that is, they can't occur at the same time), then the probability that event A *or* event B occurs is simply the sum of the two probabilities:

$$P(A \text{ or } B) = P(A) + P(B)$$
$$P(A \cup B) = P(A) + P(B)$$

If events A and B are *not* necessarily mutually exclusive (that is, they can happen at the same time), the probability that event A *or* event B occurs is as follows:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Remember: The symbol $\cup$ means union, and the symbol $\cap$ means intersection.

For example, suppose your family is planning a vacation to France or Spain and can only go to one of these destinations. The probability of choosing France is $\frac{1}{2}$ and the probability of choosing Spain is also $\frac{1}{2}$.

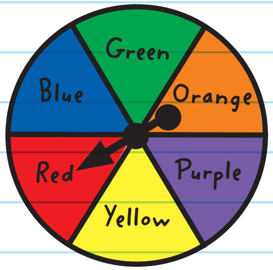
The probability of choosing either France or Spain is 1.

$$P(A \cup B) = \frac{1}{2} + \frac{1}{2} - 0$$

$$P(A \cup B) = 1$$

Think: $P(A \cap B) = 0$. Why? Your family can only go to one destination for a vacation!

EXAMPLE: A spinner has 6 equal sectors colored green, orange, purple, yellow, red, and blue. What is the probability of the spinner landing on orange or blue?



Event A = orange *OR* **Event B** = blue

$P(A) = \frac{1}{6}$ $P(B) = \frac{1}{6}$

Use the addition probability rule.

$P(A \cup B) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

Think: $P(A \cap B) = 0$.
Why? A spin can land on only one color.

So, the probability of landing on orange or blue is $\frac{1}{3}$.

EXAMPLE: Edward rolls a 6-sided number cube. What is the probability that he rolls an odd number or a number less than or equal to 3?

Let **event A** be rolling an odd number, and let **event B** be rolling a number less than or equal to 3. These events are NOT mutually exclusive. For example, the number 1 is in both events A and B.

Step 1: Determine the sample space for the given experiment. In other words, what are all the possible outcomes when rolling a number cube?

Sample space: {1, 2, 3, 4, 5, 6}

Notice that there are 6 possible outcomes.

Now let's list the outcomes in each event.

Event A (rolling an odd number): {1, 3, 5}

$P(A) = \frac{3}{6} = \frac{1}{2}$

Event B (rolling a number less than or equal to 3): {1, 2, 3}

$P(B) = \frac{3}{6} = \frac{1}{2}$

Next, highlight from each event the outcomes that are both an odd number AND a number less than or equal to 3.

Event A (rolling an odd number): {1, 3, 5}

Event B (rolling a number less than or equal to 3): {1, 2, 3}

So, $A \cap B = \{1, 3\}$, and therefore $P(A \cap B) = \frac{2}{6} = \frac{1}{3}$.

Step 2: Use the addition probability rule.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{3}$$

$$P(A \cup B) = \frac{2}{3}$$

Therefore, the probability that Edward rolls an odd number or a number less than or equal to 3 is $\frac{2}{3}$.

CHECK YOUR KNOWLEDGE

1. A produce grocer is unpacking 6 crates, each with a different type of apple: Red Delicious, Granny Smith, Honeycrisp, Gala, Fuji, and Ginger Gold. What is the probability that the grocer unpacks the Honeycrisp apples first? Assume that the grocer is equally likely to choose each crate.

2. The probability that a person inside a convenience store will purchase a drink is $\frac{3}{5}$. What is the probability that the person will not purchase a drink?

3. A bag contains 12 blue marbles, 8 purple marbles, 10 orange marbles, and 14 black marbles. A marble is drawn, then it is replaced and a second marble is drawn. What is the probability of drawing a purple marble and then an orange marble?

4. Mandy creates a set of 26 cards, each with a different letter of the alphabet, and places them all in a bucket. If Mandy's niece Riley draws one card from the bucket and then draws another card without replacing the first one, what is the probability that Riley picks first a card with a consonant and then a card with a vowel? The letter *Y* should be considered a consonant.

For questions 5 through 7, use the spinner below.

5. A game at the annual school fair requires the participant to spin the spinner. If the spinner lands on a composite number (a number with more than 2 factors), the participant wins a prize. What is the probability of the spinner landing on a composite number?

6. What is the probability of the spinner landing on a sector that is orange or red?

7. What is the probability of the spinner landing on a prime number or a number greater than 15?

ANSWERS

CHECK YOUR ANSWERS

1. $\frac{1}{6}$

2. $\frac{2}{5}$

3. $\frac{5}{121}$

4. P(consonant and vowel) =
$P(A) \cdot P(B) = \frac{21}{26} \cdot \frac{5}{25} = \frac{105}{650} = \frac{21}{130}$

5. $\frac{24}{37}$

6. P(orange or red) = $P(A) + P(B) = \frac{8}{37} + \frac{7}{37} = \frac{15}{37}$

7. P(prime number or number greater than 15) =
$P(A) + P(B) - P(A \cap B) = \frac{11}{37} + \frac{21}{37} - \frac{5}{37} = \frac{27}{37}$

CONDITIONAL PROBABILITY

CONDITIONAL PROBABILITY refers to the likelihood of one event occurring given that another event has already occurred.

For example, say that you have a bag with 7 marbles, 3 white and 4 black. You draw 1 marble from the bag at random, and then you draw another, without replacing the first one. What is the probability that the second marble you draw is black, given that the first marble you drew was white?

Since the first marble was white, there are 6 marbles left in the bag and 4 are black. So, the probability is $\frac{4}{6} = \frac{2}{3}$.

Consider two events where the second event depends on the first event. If A is the first event and B is the second event, then we may state the conditional probability as the probability of B given A, and we write $P(B|A)$.

The vertical line is read "given."

We can find the conditional probability of event B given event A using the formula

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Equivalently,

$$P(A \cap B) = P(A) \cdot P(B|A).$$

EXAMPLE: Meghan has 40 cards, each one with a different number written on it from 1 through 40. While playing a game in the class, a student selects one card, doesn't replace it, and then selects another card. What is the probability of selecting first a card with an odd number written on it and then a card with an even number written on it?

Given that each of the 40 cards has a different number written on it, from 1 through 40, you know that half of those cards (20) have an odd number and the other half (20) have an even number.

Event A: Selecting a card with an odd number.

Event B: Selecting a card with an even number after selecting a card with an odd number.

Think: Since the first card is not replaced, event B DEPENDS ON event A.

Probability of event A:

$$P(A) = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}} = \frac{20}{40}$$

Probability of event B given event A:

$$P(B|A) = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes remaining}} = \frac{20}{39}$$

(there is 1 less card)

Probability of compound event:

$$P(A \cap B) = P(A) \cdot P(B|A) = \frac{20}{40} \cdot \frac{19}{39} = \frac{19}{78}$$

So, the probability of selecting a card with an odd number followed by a card with an even number is $\frac{19}{78}$.

EXAMPLE: If two 6-sided number cubes are rolled, what is the probability that the resulting sum is 8, if it is known that exactly one of the number cubes shows a 3?

There are 36 possible outcomes when we roll two number cubes.

Possible outcomes:

NUMBER CUBE 2 \ NUMBER	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(3, 4)	(4, 5)	(3, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

In this problem, we are specifically looking at the case where the sum of the two cubes is 8, given that one of the number cubes shows a 3.

Let A be the event that *exactly* one of the number cubes shows a 3. The favorable outcomes in event A are highlighted in the table on the previous page.

$A = \{(1, 3), (2, 3), (4, 3), (5, 3), (6, 3), (3, 1), (3, 2), (3, 4), (3, 5), (3, 6)\}$.

So, there are 10 outcomes in event A, and $P(A) = \frac{10}{36} = \frac{5}{18}$.

Let B be the event that the sum of the two number cubes is 8. Then $A \cap B$ is the event that the sum of the two number cubes is 8 *and* exactly one of the number cubes shows a 3.

$A \cap B = \{(5, 3), (3, 5)\}$.

So, there are 2 outcomes in event $A \cap B$, and $P(A \cap B) = \frac{2}{36} = \frac{1}{18}$.

It follows that $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{18}}{\frac{5}{18}} = \frac{1}{5}$.

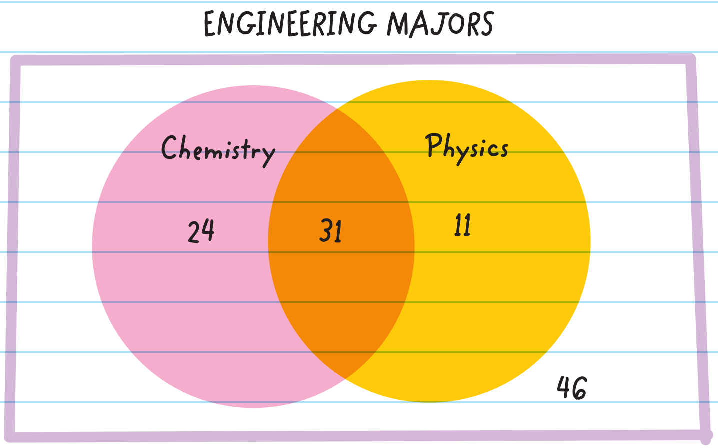
Therefore, if *exactly* one of the number cubes shows a 3, the probability that the resulting sum is 8 is $\frac{1}{5}$.

EXAMPLE: In a group of 112 engineering majors, 55 took chemistry in high school, 42 took physics in high school, and 31 took both chemistry and physics in high school. If an engineering major who took chemistry in high school is chosen at random, what is the probability they also took physics in high school?

We can use a Venn diagram to illustrate this problem.

To get the number of engineering majors who took *only* chemistry, compute $55 - 31 = 24$.

To get the number who took *only* physics, compute $42 - 31 = 11$.



To get the number who took neither chemistry nor physics, compute $112 - 24 - 31 - 11 = 46$.

Probability of an engineering major taking chemistry:

$P(A) = \frac{55}{112}$

Probability of an engineering major taking chemistry and physics:

$P(A \cap B) = \frac{31}{112}$

It follows that $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{31}{112}}{\frac{55}{112}} = \frac{31}{55}$.

Therefore, the probability that an engineering major who took chemistry in high school also took physics in high school is $\frac{31}{55}$.

CHECK YOUR KNOWLEDGE

1. A bakery receives a crate of 72 apples. In the crate, 42 apples are Granny Smith and 30 are Golden Delicious. One apple is randomly picked from the crate and not replaced, and then a second apple is picked. What is the probability that the second apple picked is Golden Delicious, given that the first apple picked was Granny Smith?

2. During a fundraising event, the cheerleading team conducted a raffle to give away two electronic devices. Each $10 donation entitled the donor to a ticket, and these tickets were distributed as either blue or red. A total of 330 blue tickets and 300 red tickets were distributed. At the homecoming game, two winners are selected by drawing two tickets in succession. What's the probability that the second ticket drawn will be red, given that the first ticket drawn was blue?

3. Gerard randomly selects 2 cards from a deck of 60 trading cards. In the deck, 40 cards show players from Team A and the other 20 cards show players from Team B. What is the probability that the first card shows a player from Team A and the second card shows a player from Team B if (A) Gerard replaces the first card before selecting the second card, and (B) Gerard does not replace the first card?

4. Sohail tosses a fair coin twice.
   A. What is the probability that the coin will land on tails both times?
   B. If the first coin toss results in tails, what is the probability that both coins will land on tails?

5. A city parks department surveys high school students to determine whether they own only a bicycle, only in-line skates, both a bicycle and in-line skates, or neither a bicycle nor in-line skates. The results of the survey are shown in the table below.

Owns only a bicycle	Owns only in-line skates	Owns both a bicycle and in-line skates	Owns neither a bicycle nor in-line skates
380	605	274	215

If one of the students who participated in the survey is chosen at random, what is the probability they own both a bicycle and in-line skates?

ANSWERS

CHECK YOUR ANSWERS

1. $\frac{30}{71}$

2. $\frac{300}{629}$

3. A. $\frac{2}{9}$

   B. $\frac{40}{177}$

4. A. $\frac{1}{4}$

   B. $\frac{1}{2}$

5. $\frac{137}{737}$

BINOMIAL THEOREM

Recall that a **BINOMIAL** is an expression with two terms. So, a binomial can be written in the form $a + b$.

The **BINOMIAL THEOREM** provides us with a formula for raising a binomial to a positive integer power.

In other words, it allows us to write $(a + b)^n$ in an expanded form.

For example, the expanded form of $(a + b)^2$ is

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$$

We'll explore the Binomial Theorem in a moment. First, let's look at Pascal's triangle, which has an important relationship with the Binomial Theorem.

PASCAL'S TRIANGLE

In general, **PASCAL'S TRIANGLE** can be used to find the coefficients in the expansion of the expression $(a + b)^n$ for any positive integer n.

Each row of Pascal's triangle has one more number than the preceding row.

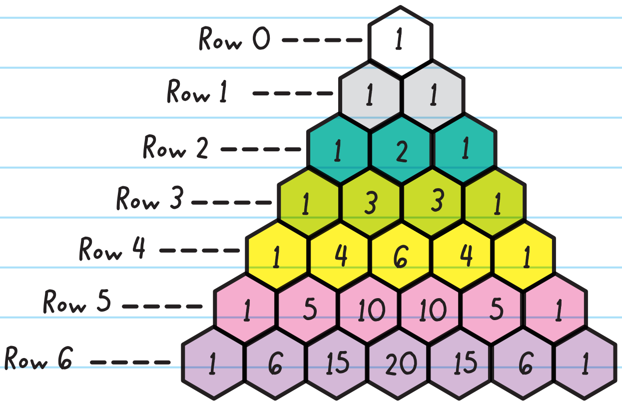
- Row 0 consists of just the number 1, and Row 1 consists of 1 followed by another 1.
- The numbers in each subsequent row can be found by adding pairs of adjacent numbers in the preceding row (except for the left and right numbers, which are always 1).
- So, for Row 2, since $1 + 1 = 2$, we have a 2 in the middle with a 1 at each end. It looks like 1 2 1.

These numbers help us write the expanded form of $(a + b)^2$:
$1a^2 + 2ab + 1b^2 = a^2 + 2ab + b^2$

- Row 3 is 1 3 3 1.

So, $(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3 = a^3 + 3a^2b + 3ab^2 + b^3$

This pattern continues forever.



PASCAL'S TRIANGLE

Notice the following:

$1 + 2 + 1 = 4 = 2^2$

$1 + 3 + 3 + 1 = 8 = 2^3$

$1 + 4 + 6 + 4 + 1 = 16 = 2^4$

. .
. .
. .

Sum of each row = $2^{(\text{Number of the row})}$

$1 + 7 + 21 + 35 + 35 + 21 + 7 + 1$

$= 128 = 2^7$

EXAMPLE: Use Pascal's triangle to expand $(a + b)^4$ and $(a + b)^5$.

$(a + b)^4$:

Since $n = 4$, use ROW 4 of Pascal's triangle to find the coefficients of the expanded polynomial: 1, 4, 6, 4, 1.

Start with a^4b^0, and as you go from one term to the next, decrease the power of a by 1 and increase the power of b by 1.

The expansion of $(a + b)^4$ is as follows:

$(a + b)^4 = 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4$

$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Therefore, the expanded form of $(a + b)^4$ is $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a + b)^5$:

Since $n = 5$, use ROW 5 of Pascal's triangle to find the coefficients of the expanded polynomial: 1, 5, 10, 10, 5, 1.

Start with a^5b^0, and as you go from one term to the next, decrease the power of a by 1 and increase the power of b by 1.

The expansion of $(a + b)^5$ is as follows:

$(a + b)^5 = 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5$

$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Therefore, the expanded form of $(a + b)^5$ is $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

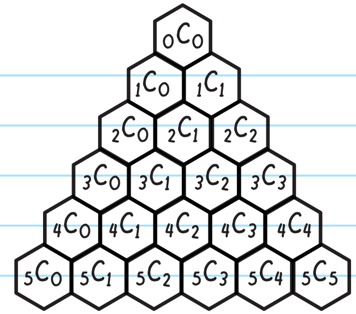
BINOMIAL THEOREM

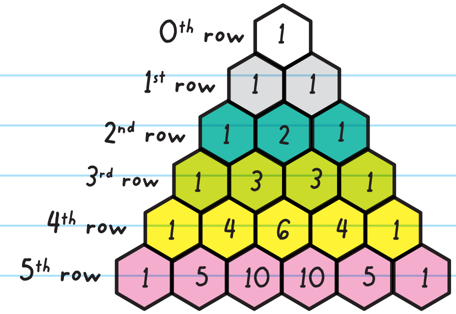
The **BINOMIAL THEOREM** says: for any positive integer n and any real numbers a and b,

$(a + b)^n =$

${}_nC_0a^nb^0 + {}_nC_1a^{(n-1)}b^1 + {}_nC_2a^{(n-2)}b^2 + \ldots + {}_nC_{(n-1)}a^1b^{(n-1)} + {}_nC_na^0b^n$.

Notice that the coefficients of $(a + b)^n$ given in the Binomial Theorem are just the numbers in the nth row of Pascal's triangle.





Now let's use the Binomial Theorem to compute $(a + b)^n$ for $n = 2$ and $n = 3$.

For $n = 2$:

$(a + b)^2 = {}_2C_0a^2b^0 + {}_2C_1a^1b^1 + {}_2C_2a^0b^2$

$= 1a^2 + 2ab + 1b^2$

$= a^2 + 2ab + b^2$

Think: The coefficients ${}_2C_0$, ${}_2C_1$, and ${}_2C_2$ are just the numbers in Row 2 of Pascal's triangle: 1 2 1.

So, for $n = 2$, the expansion of $(a + b)^2$ is $a^2 + 2ab + b^2$.

For $n = 3$:

$(a + b)^3 = {}_3C_0\,a^3b^0 + {}_3C_1\,a^2b^1 + {}_3C_2a^1b^2 + {}_3C_3a^0b^3$

$= 1a^3 + 3a^2b + 3ab^2 + 1b^3$

$= a^3 + 3a^2b + 3ab^2 + b^3$

Think: The coefficients ${}_3C_0$, ${}_3C_1$, ${}_3C_2$, and ${}_3C_3$ are just the numbers in Row 3 of Pascal's triangle: 1 3 3 1.

So, for $n = 3$, the expansion of $(a + b)^3$ is $a^3 + 3a^2b + 3ab^2 + b^3$.

EXAMPLE: Compute the following expressions.

1. $(x + 1)^4$

Since $n = 4$, use ROW 4 of Pascal's triangle to find the coefficients of the expanded polynomial: 1, 4, 6, 4, 1.

$(x + 1)^4 = 1x^4 1^0 + 4x^3 1^1 + 6x^2 1^2 + 4x^1 1^3 + 1x^0 1^4$

$= x^4 + 4x^3 + 6x^2 + 4x + 1$

So, $(x + 1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$.

2. $(2x + 3y)^3$

Since $n = 3$, use ROW 3 of Pascal's triangle to find the coefficients of the expanded polynomial: 1, 3, 3, 1.

$(2x + 3y)^3 = 1(2x)^3(3y)^0 + 3(2x)^2(3y)^1 + 3(2x)^1(3y)^2 + 1(2x)^0(3y)^3$

$= 8x^3 + 3 \cdot 4x^2 \cdot 3y + 3 \cdot 2x \cdot 9y^2 + 27y^3$

$= 8x^3 + 36x^2y + 54xy^2 + 27y^3$

So, $(2x + 3y)^3 = 8x^3 + 36x^2y + 54xy^2 + 27y^3$.

3. $(\frac{2}{x} + 1)^3$

Since $n = 3$, use ROW 3 of Pascal's triangle to find the coefficients of the expanded polynomial: 1, 3, 3, 1.

$\left(\frac{2}{x} + 1\right)^3 = 1\left(\frac{2}{x}\right)^3 \cdot 1^0 + 3\left(\frac{2}{x}\right)^2 \cdot 1^1 + 3\left(\frac{2}{x}\right)^1 \cdot 1^2 + 1\left(\frac{2}{x}\right)^0 \cdot 1^3$

$= \frac{8}{x^3} + 3 \cdot \frac{4}{x^2} + 3 \cdot \frac{2}{x} + 1$

$= \frac{8}{x^3} + \frac{12}{x^2} + \frac{6}{x} + 1$

So, $\left(\frac{2}{x} + 1\right)^3 = \frac{8}{x^3} + \frac{12}{x^2} + \frac{6}{x} + 1$.

CHECK YOUR KNOWLEDGE

For questions 1 through 6, compute the expressions using the Binomial Theorem.

1. $(3x + 2y)^2$
2. $\left(\frac{1}{a} + 2b\right)^3$
3. $(5x - y)^3$
4. $\left(2a + \frac{2}{b}\right)^4$
5. $(x^3 + 2y^3)^3$
6. $(d - 10)^5$

ANSWERS

CHECK YOUR ANSWERS

1. $9x^2 + 12xy + 4y^2$

2. $\frac{1}{a^3} + \frac{6b}{a^2} + \frac{12b^2}{a} + 8b^3$

3. $125x^3 - 75x^2y + 15xy^2 - y^3$

4. $16a^4 + \frac{64a^3}{b} + \frac{96a^2}{b^2} + \frac{64a}{b^3} + \frac{16}{b^4}$

5. $x^9 + 6x^6y^3 + 12x^3y^6 + 8y^9$

6. $d^5 - 50d^4 + 1{,}000d^3 - 10{,}000d^2 + 50{,}000d - 100{,}000$

REPRESENTING DATA

DATA is a collection of facts in the form of numbers, words, or descriptions. **STATISTICS** is the organization, presentation, and study of data.

TYPES OF DATA

There are two types of data: quantitative data and qualitative data.

QUANTITATIVE DATA: Information that is given in *numbers*. Usually this information can be counted or measured.

QUALITATIVE DATA: Information given that *describes* something. Usually this is information that can be observed, such as appearances, textures, smells, and tastes.

Quantitative and qualitative data can be collected, interpreted, and summarized in words, graphs, and images.

COLLECTING DATA

A **STATISTICAL QUESTION** is a question that anticipates having many different responses.

"How tall am I?"

This question has only one answer. It is not a statistical question.

"How tall are the students on the basketball team?"

This question has more than one answer, so it is a statistical question.

VARIABILITY describes how spread out or closely clustered a collection of data is.

The answers to a statistical question can have **HIGH VARIABILITY** (they are very spread out) or **LOW VARIABILITY** (they are closely clustered).

SAMPLING

Sometimes we can gather data from every member in a group. However, most of the time that's not feasible. Therefore, we use a **SAMPLE**, a small part of a larger group to estimate characteristics of the whole group.

EXAMPLE: There are 862 students entering a new agricultural high school in the fall semester. The principal, Mr. Thomas, is curious to know how many incoming students would be interested in joining a summer community gardening project. To gather data, Mr. Thomas randomly surveys 100 students and determines that 82 of them are interested in joining this summer project. Based on this random sample, approximately how many incoming high school students would likely be interested in participating in the project? Round the answer to the nearest student.

Since 82 of 100 members were interested, it is likely that approximately $\frac{82}{100} = \frac{41}{50}$ of the students will be interested.

$$862 \cdot \frac{41}{50} = \frac{17{,}671}{25} = 706.84$$

Therefore, approximately 707 students would likely be interested in participating in the summer gardening project.

DISPLAYING AND ANALYZING DATA

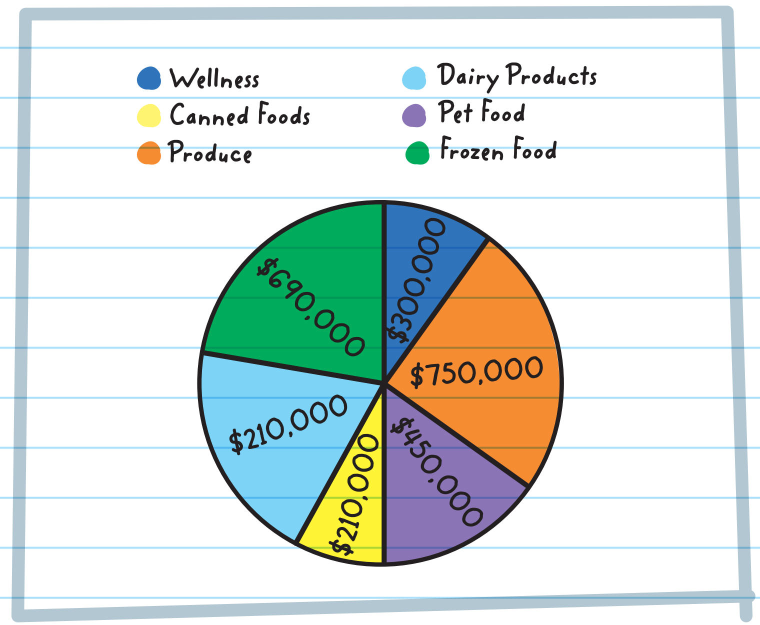
Tables are used to present data in list form. But we can also represent data visually with graphs and diagrams.

Circle Graphs

A **CIRCLE GRAPH** is divided into sectors, with each sector representing a portion in a collection of data.

This circle graph visually represents the revenue amassed from sales made in different departments of a grocery store.

GROCERY STORE SALES (JANUARY)



TWO-WAY TABLES

A **TWO-WAY TABLE** has rows and columns. It shows two or more categories of data collected from the same data group.

The two-way table below shows data collected from 153 math students pertaining to their average in math and whether they speak a world language.

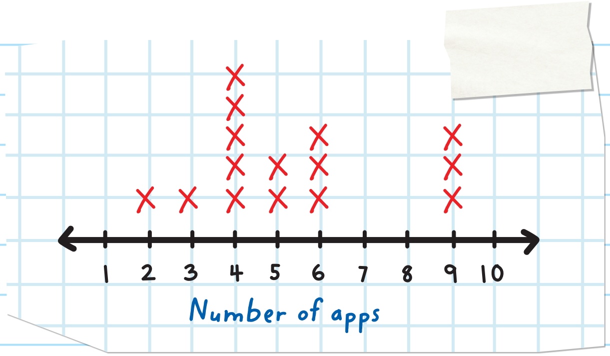
	Speak a World Language	Do Not Speak a World Language	Total
Math Average ≥ 90	70	20	90
Math Average < 90	15	48	63
Total	85	68	153

LINE PLOTS

A **LINE PLOT** provides a simple way to display the frequency of data. It displays data by placing X's above numbers on a number line.

The line plot shown displays the number of apps a selection of students have on their cell phones.

NUMBER OF APPS ON CELL PHONE



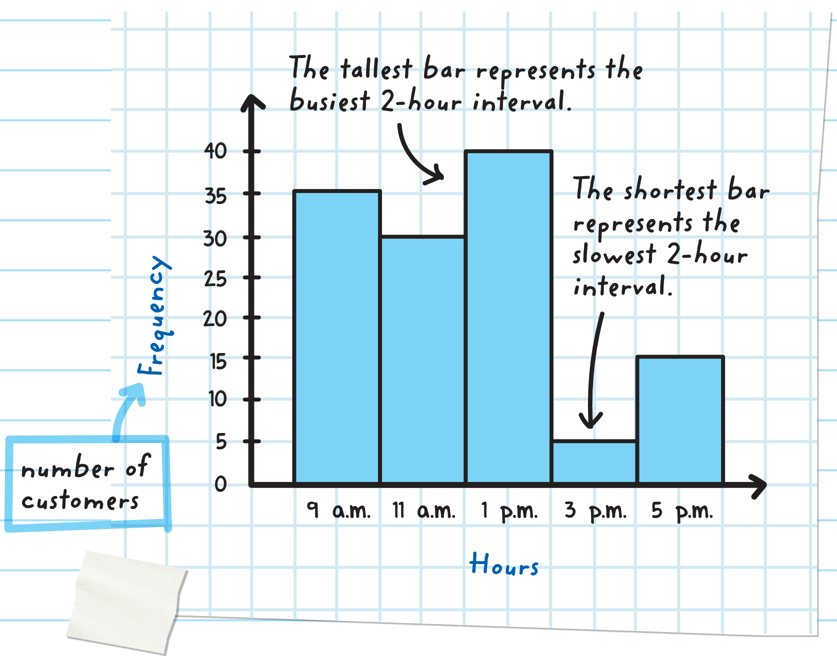
HISTOGRAMS

A **HISTOGRAM** is a graph that shows the frequency of data within intervals. It looks like a bar graph, but unlike a bar graph, there are no gaps between the vertical or horizontal bars unless an interval has a frequency of 0.

The histogram on the following page shows the number of customers who called customer service in a 10-hour period. The histogram is divided into 2-hour intervals.

We always assume that the left endpoint of an interval is included in the interval and the right endpoint is not.

PHONE CALLS TO CUSTOMER SERVICE



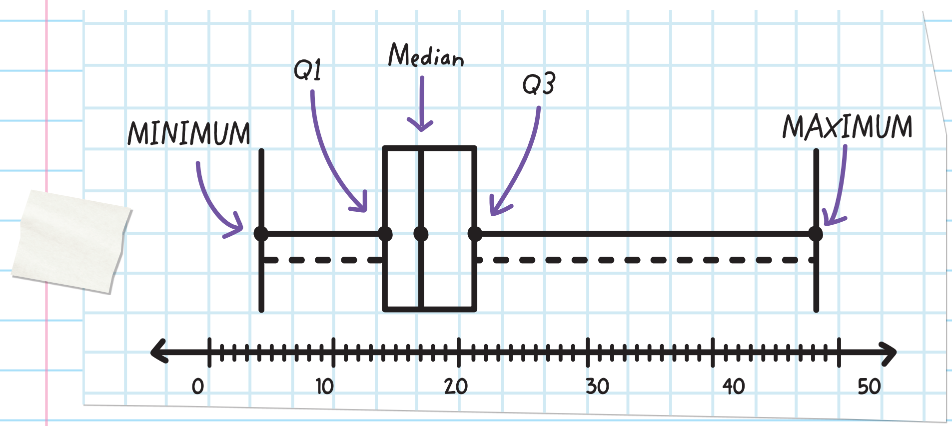
From the graph, we can see the following:

- 35 customers called between 9 a.m. and 11 a.m.
- 15 customers called between 3 p.m. and 5 p.m.

BOX PLOTS

A **BOX PLOT** is a diagram that shows how data is distributed. It does not show all the data. Instead, it summarizes the spread of the data.

The data is displayed along a number line and is split into **QUARTILES** (quarters). The **MEDIAN** of the data (the middle number when the data is arranged in increasing order) separates the data into halves. The quartiles are values that divide the data into fourths. The median of the lower half is the lower quartile of the data and is represented by **Q1**. The median of the upper half is the upper quartile of the data and is represented by **Q3**. The size of each section indicates the variability of the data.



The box plot shows:

25% of the data was above 21.
Q3 up to the maximum

25% of the data was between 17 and 21.
the median up to Q3

25% of the data was between 14 and 17.
Q1 up to the median

25% of the data was below 14.
Q1 down to the minimum

Notice that in the given box plot, the right-hand portion of the box is wider than the left-hand portion. When a box graph is not evenly divided in half, the data is said to be SKEWED.

If the box plot has a wider right side, the graph is described as **skewed right**.

If the box plot has a wider left side, the graph is described as **skewed left**.

If the box plot is evenly divided, the graph is described as **symmetrical**.

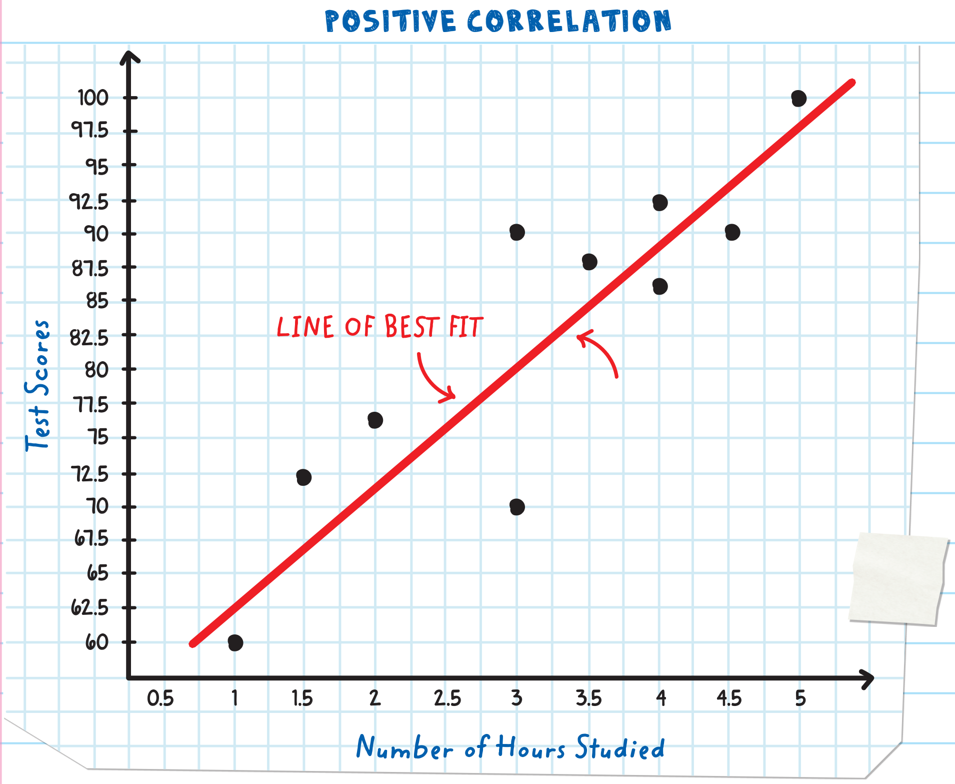
SCATTER PLOTS

A SCATTER PLOT is a graph that compares two related collections of data on a coordinate plane. Scatter plots graph data as ORDERED PAIRS.

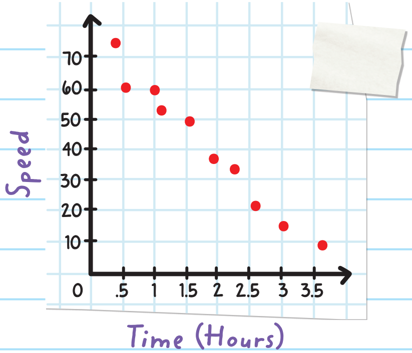
By graphing two sets of data on a scatter plot, we can tell if there is any relationship or CORRELATION between them. In the scatter plot on the next page, the relationship is between the number of hours studied and test scores. Notice that the scores go up as the hours of studying go up. Therefore, this scatter plot shows that there is a relationship between test scores and studying. This relationship is known as a POSITIVE CORREALTION.

A line can be drawn on the graph that roughly describes the relationship between the two sets of data. This line is called the LINE OF BEST FIT. The line of best fit represents the general trend of the data. It is a good indicator of how the points are related to one another.

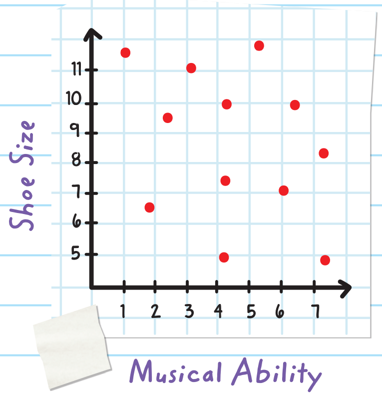
POSITIVE CORRELATION: As one variable increases, the other variable increases as well.



NEGATIVE CORRELATION: As one variable increases, the other variable decreases.

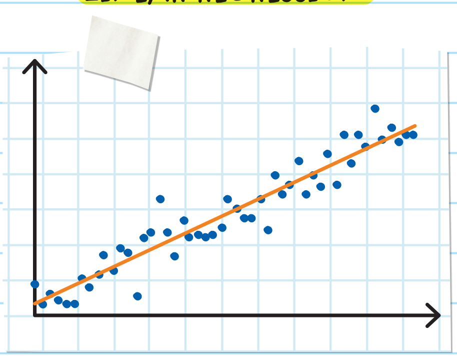


NO CORRELATION: The values have no relationship.

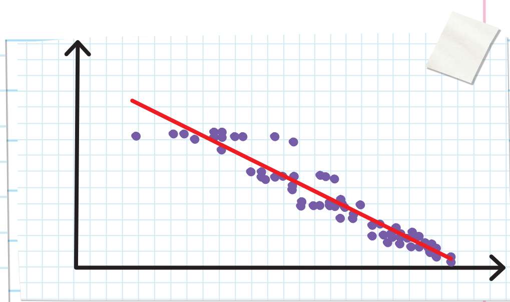


The process of using a line of best fit to summarize data and make future predictions is called **LINEAR REGRESSION**.

LINEAR REGRESSION

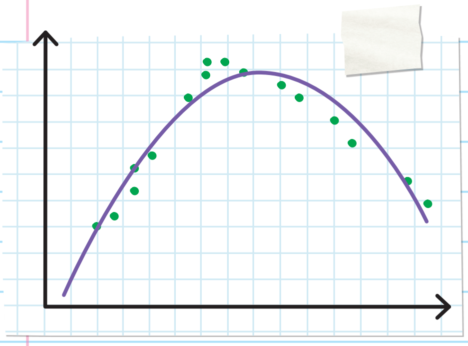


LINEAR REGRESSION

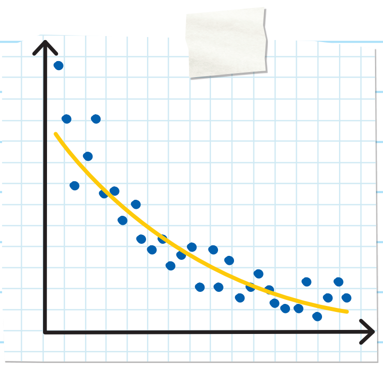


Sometimes a line might not be very useful for summarizing the data, but another kind of function might work. In this case, we might use **QUADRATIC REGRESSION**, or **EXPONENTIAL REGRESSION**.

QUADRATIC REGRESSION



EXPONENTIAL REGRESSION

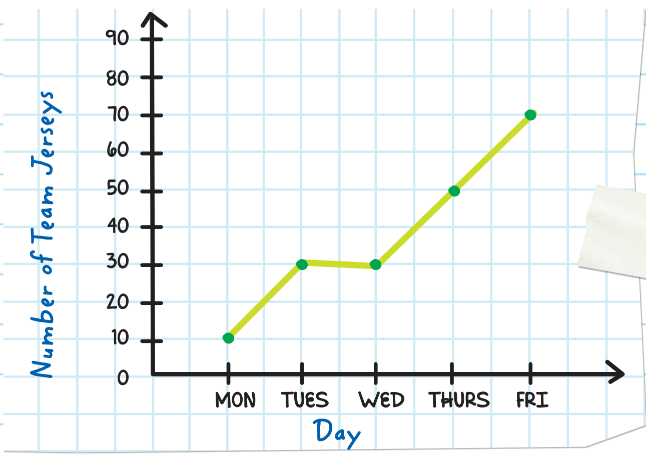


LINE GRAPHS

A **LINE GRAPH** is a graph that can be used to show a change in data over time.

For example, a store can use a line graph to keep a record of how many team jerseys it sells each day of the week.

TEAM JERSEYS SOLD PER DAY



Based on the line graph, it appears that the store sells the least jerseys at the beginning of the week and the most jerseys at the end of the week.

STEM-AND-LEAF PLOTS

A **STEM-AND-LEAF-PLOT** organizes data from least to greatest using place values.

For example, an online store can look at the reward points earned by its first 35 customers based on the amount of their purchases.

REWARD POINTS EARNED				
86	77	91	60	55
76	92	47	88	67
23	59	72	75	83
77	68	82	97	89
81	75	74	39	67
79	83	70	78	91
68	49	56	94	81

STEM	LEAF
2	3
3	9
4	7 9
5	5 6 9
6	0 7 7 8 8
7	0 2 4 5 5 6 7 7 8 9
8	1 1 2 3 3 6 8 9
9	1 1 2 4 7

KEY: 2|3 = 23

Observe how the **STEMS** are all the possible tens places that appear in the data and the **LEAVES** are all the ones places that appear in the data. Each possible stem is listed just once, whereas every leaf is listed, even if there is repetition.

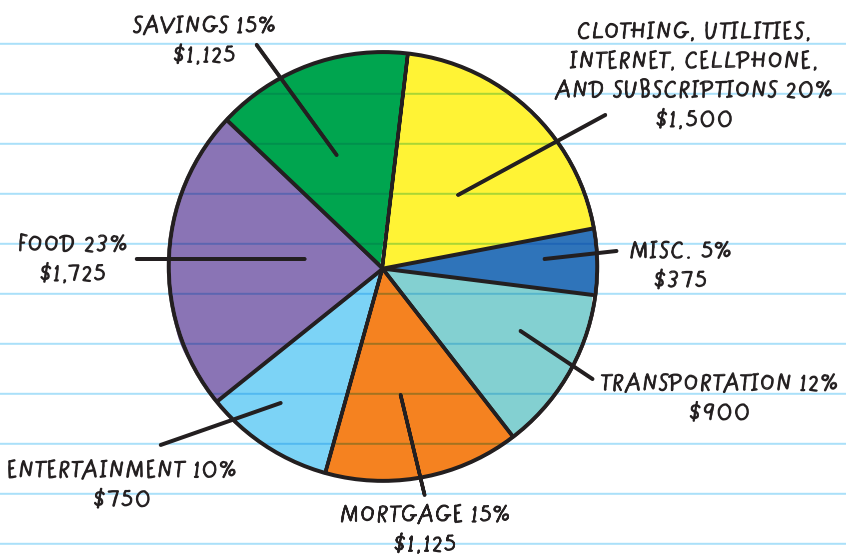
Also, observe that the rewards **CLUSTER** around the 60 to 90 range.

OUTLIERS

A value that is significantly lower or higher than the other values in a data set is called an **OUTLIER**. An outlier can throw off the average of a collection of data and give a skewed portrayal of the data.

CHECK YOUR KNOWLEDGE

1. The Bannor family uses their net income to create a monthly budget, as shown in the circle graph below.



Answer the following questions based on the circle graph.

A. Which category has the *greatest* allocation of income? Which category has the *least* allocation of income?

B. To which two categories does the Bannor family allocate the same amount of money? What is the total percentage of that income allocation?

C. Does the Bannor family allocate more money to their mortgage and savings than they do to clothing, utilities, internet, cell phone service, and subscriptions? Explain how you came up with your answer.

D. What is the Bannor family's monthly net income?

2. The two-way table below represents the results of a survey given to high school students currently taking environmental science.

Answer the following questions based on the table.

	Owns ski gear	Owns swimming gear	Total
Likes the winter season best	119	57	176
Likes the summer season best	32	86	118
Total	151	143	294

A. How many students like the winter season best and own swimming gear?

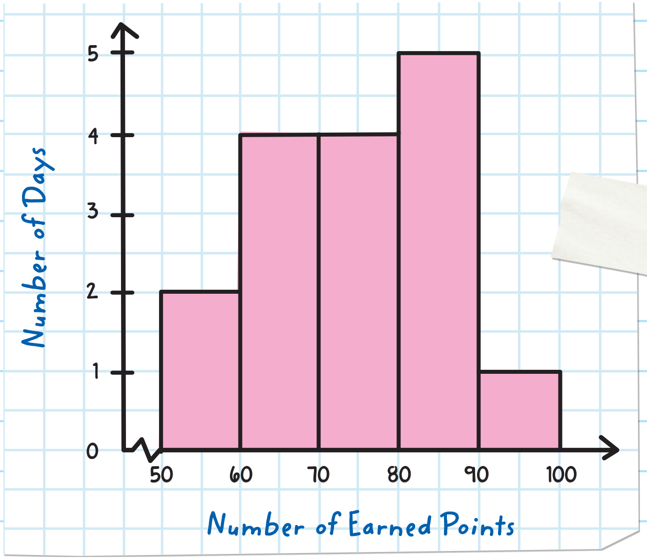
B. How many students like the summer season best and own ski gear?

C. How many students like the winter season best and own ski gear?

3. The owner of an audiobook app sends out a survey asking select customers how many audiobooks they downloaded in the past three months. The responses were 5, 3, 6, 4, 9, 9, 4, 8, 5, 10, 4, 3, 2, 1, 7, 4, 3, 2, 3, 2, and 4. Create a line plot from the data collected by the app's owner. Then answer the following question: Does this data contain any outliers?

4. Joshua's fitness app allows him to earn points every time he performs various modes of exercise. The given histogram shows the points Joshua earned over the span of 16 days.

JOSHUA'S FITNESS POINTS



Answer the following questions based on the histogram.

A. On how many days did Joshua earn between 60 and 79 points?

B. On how many days did Joshua earn at least 80 points?

C. Which interval shows five days of earned points?

5. A university coach records the heights of all players on the volleyball team.

Volleyball Player Heights (in inches)

59, 66, 66, 68, 68, 69, 69, 69, 69, 70, 70, 70, 70, 70, 71, 72, 72, 72, 72, 72, 73, 73, 73, 74, 74, 75, 76, 76, 76

Create a box plot of the given data. Then answer the following questions.

A. Where is the lower quartile located?

B. Where is the upper quartile located?

6. The high school basketball team records the number of people attending their weekday games in the stem-and-leaf plot below.

Answer the following questions based on the stem-and-leaf plot.

STEM	LEAF
10	3
11	3
12	1 5 4 5 0 4
13	9 0 0 5 0
14	0 0 0
15	0 5 7 0 0
16	0
17	0
18	
19	5
20	0

KEY: 10|3 = 103

A. Which stem has the most attendance recorded?

B. How many basketball games had less than 160 people in attendance?

CHECK YOUR ANSWERS

1. **A.** greatest allocation = food;
least allocation = misc

B. savings and mortgage; 15% + 15% = 30%

C. Yes. 30% of the family's income is allocated to savings and mortgage, and only 20% is allocated to clothing, utilities, internet, cell phone service, and subscriptions.

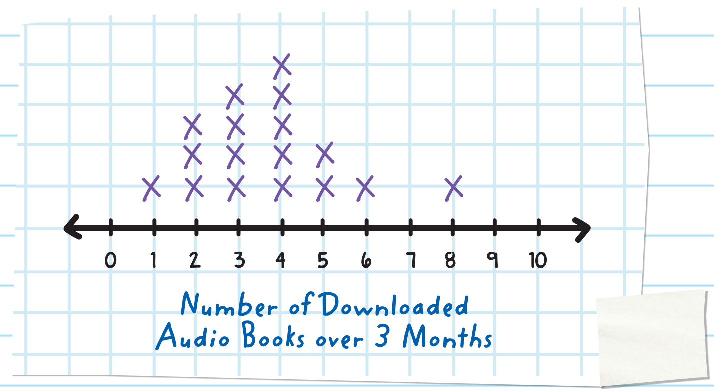
D. 7,500

2. **A.** 57

B. 32

C. 119

3. AUDIO BOOK APP SURVEY



The collection of data contains no outliers.

MORE ANSWERS

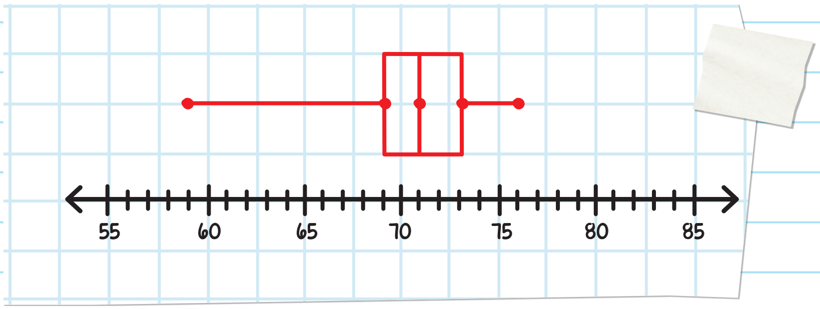
4. **A.** 8

B. 6

C. The interval from 80 to 89 points.

5.

VOLLEYBALL PLAYER HEIGHTS (IN INCHES)



A. 69

B. 73

6. **A.** 12

B. 21

MEASURES OF CENTRAL TENDENCY AND VARIANCE

MEASURES OF CENTRAL TENDENCY

Information about a data set can be conveyed by using **MEASURES OF CENTRAL TENDENCY**. A measure of central tendency is a single number that is used to describe a whole collection of data.

The three most common measures of central tendency are the mean, median, and mode.

Mean

The **MEAN** (also called the average) is one of the most often used measures of central tendency.

To calculate the mean, add up all the data, then divide the sum by the number of addends. The mean is most useful when the data values are close together.

For example, suppose that Darren took 6 tests in math and received scores of 78, 90, 80, 85, 85, and 92. To find the mean (average), add all the test scores and divide them by the number of scores (6).

$$\frac{78+90+80+85+85+92}{6} = \frac{510}{6} = 85$$

So, Darren's average math test score is 85.

Median

The **MEDIAN** is the middle number when the data is arranged in increasing order.

The greatest value in a data set is called the **MAXIMUM**.

The lowest value is called the **MINIMUM**.

The middle number (when the values are listed in order) is called the **MEDIAN**.

When there is *no* middle number, find the mean of the two values in the middle by adding them together and then dividing by 2.

For example, let's go back to Darren's test scores:

78, 90, 80, 85, 85, 92

The maximum of the data or highest test score is 92, and the minimum of the data or lowest test score is 78.

To find the median, we first need to rewrite the data in increasing order:

78, 80, 85, 85, 90, 92

The middle two terms are 85 and 85. Therefore, the median is also 85.

So, Darren's median mathematics test score is 85. His maximum test score is 92, and his minimum test score is 78.

Let's look at another example. Here's the data:

6, 3, 3, 10

First, rewrite the data in increasing order:

3, 3, 6, 10

Next, identify the minimum and maximum values.

The minimum is 3. The maximum is 10.

Then find the median. Since there is no middle term, calculate the sum of the two middle terms and divide the result by 2.

$3 + 6 = 9$

$9 \div 2 = 4.5$

So, the median is 4.5.

Mode

The **MODE** is the value that occurs most often in a collection of data. There can be one mode, more than one mode, or no modes at all. (If no values are repeated, we say that there is no mode.)

Let's go back to Darren's math test scores:

78, 90, 80, 85, 85, 92

In this data, only one test score repeats: 85. So, the mode of Darren's math test scores is 85.

EXAMPLE: To anticipate future resource requirements for its media center, a university tracks the number of students who attend the center daily throughout the fall semester. The table below lists the in-person usage totals.

DAY	NUMBER OF STUDENTS
Monday	1,798
Tuesday	2,153
Wednesday	2,187
Thursday	1,964
Friday	983
Saturday	983
Sunday	1,006

The head of planning and development at the university wants to use measures of central tendency to assess the data. Find the mean, median, minimum, maximum, and mode for this data.

Step 1: Find the mean (average).

Add up all the data, and then divide the sum by the number of addends.

$$\text{mean} = \frac{1{,}798 + 2{,}153 + 2{,}187 + 1{,}964 + 983 + 983 + 1{,}006}{7}$$

$$\text{mean} = \frac{11{,}074}{7}$$

$$\text{mean} = 1{,}582$$

Step 2: Find the median, minimum, and maximum.

Place the data in order from least to greatest.

minimum ↙ 983, 983, 1,006, median ↙ 1,798, 1,964, 2,153, maximum ↙ 2,187

Step 3: Find the mode.

Only one number repeats: 983. Therefore, the mode is 983.

MEASURES OF VARIATION

We can also use **MEASURES OF VARIATION** to provide information about data. Measures of variation describe how the values in a collection of data vary. The simplest measure of variation is **RANGE**. The range of a collection of data is the difference between the maximum and minimum values. The range shows how "spread out" the data is.

For example, let's revisit Darren's math test scores:

78, 80, 85, 85, 90, 92

The range is 92 − 78 = 14.

Think: We organize the scores from *least* to *greatest* to determine the range.

So, the range of Darren's math test scores is 14.

VARIANCE and **STANDARD DEVIATION** measure how spread out the data is from the mean.

To compute the **variance** of a set of data, follow these steps.

1. Find the mean of the data. Let's call the mean x.
2. For each piece of data y, find the difference $y - x$.
3. Square each of the numbers found in Step 2.
4. Average the numbers found in Step 3.

To compute the **standard deviation**:

5. Take the square root of the number found in Step 4.

EXAMPLE: A job-seeking app has a function that analyzes entry-level salaries for civil engineering positions across eight major engineering companies. This analysis serves to inform its users about the extent of income variability as they apply to job postings. Refer to the table below for detailed salary data.

Calculate the standard deviation of these entry-level salaries to the nearest dollar to gauge the extent of salary variation. Then determine the range to further assess the salary spread.

CIVIL ENGINEER ENTRY-LEVEL SALARIES			
\$68,000	\$82,000	\$69,200	\$88,000
\$76,000	\$80,000	\$90,000	\$79,000

Follow the steps given to find the variance.

Step 1: Find the mean of the data.

mean =

$$\frac{68{,}000 + 82{,}000 + 69{,}200 + 88{,}000}{8} + \frac{76{,}000 + 80{,}000 + 90{,}000 + 79{,}000}{8}$$

$$\text{mean} = \frac{632{,}200}{8}$$

mean = 79,025

Let x be the mean, so that x = 79,025.

Step 2: For each piece of data (salary) y, find the difference $y - x$.

SALARY (y)	$y - x$
$68,000	68,000 − 79,025 = −11,025
$82,000	82,000 − 79,025 = 2,975
$69,200	69,200 − 79,025 = −9,825
$88,000	88,000 − 79,025 = 8,975
$76,000	76,000 − 79,025 = −3,025
$80,000	80,000 − 79,025 = 975
$90,000	90,000 − 79,025 = 10,975
$79,000	79,000 − 79,025 = −25

Step 3: Square each number found in Step 2.

DIFFERENCE	SQUARE EACH DIFFERENCE
−11,025	$(-11{,}025)^2 = 121{,}550{,}625$
2,975	$2{,}975^2 = 8{,}850{,}625$
−9,825	$(-9{,}825)^2 = 96{,}530{,}625$
8,975	$8{,}975^2 = 80{,}550{,}625$
−3,025	$(-3{,}025)^2 = 9{,}150{,}625$
975	$975^2 = 950{,}625$
10,975	$10{,}975^2 = 120{,}450{,}625$
−25	$(-25)^2 = 625$

Step 4: Average the numbers found in Step 3.

$$\text{mean} = \frac{121{,}550{,}625 + 8{,}850{,}625 + 96{,}530{,}625 + 80{,}550{,}625}{8} + \frac{9{,}150{,}625 + 950{,}625 + 120{,}450{,}625 + 625}{8}$$

$$\text{mean} = \frac{438{,}035{,}000}{8}$$

$$\text{mean} = 54{,}754{,}375$$

Step 5: Take the square root of the number found in Step 4. Round to the nearest whole number.

$$\sqrt{54{,}754{,}375} \approx 7{,}400$$

So, the standard deviation of the entry-level salaries is approximately 7,400.

This means that most of the salaries (approximately 68% of them) are within $7,400 of the mean.

Now calculate the range.

The greatest salary is $90,000. The lowest salary is $68,000.

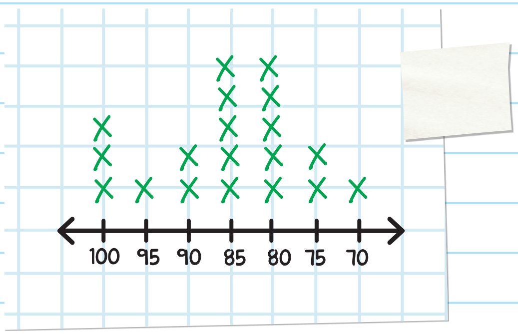
The range is 90,000−68,000 = 22,000.

So, the range of the entry-level salaries is $22,000.

CHECK YOUR KNOWLEDGE

1. The given line plot displays the final exam scores for Ms. Carly's algebra class.

ALGEBRA FINAL EXAM TEST SCORES



A. Find the mean, minimum, maximum, median, mode, and range of the final exam scores. Round calculations to the nearest hundredth.

B. Does this collection of data contain any outliers? If so, what are the outliers?

2. A high school's environmental group collects and recycles bottles and cans to keep their community clean and raise money for future endeavors. The table below lists the number of bottles and cans the group collects over a period of eight weeks.

Week 1	1,436	Week 5	1,135
Week 2	1,008	Week 6	1,135
Week 3	1,207	Week 7	1,923
Week 4	1,421	Week 8	1,246

A. For this collection of data, find the mean, minimum, maximum, median, mode, and range. Round all calculations to the nearest bottle/can.

B. Does this collection of data contain any outliers? If so, what are the outliers?

3. In Ms. Val's science class, students record from April 1 to April 5 the temperature of their school's outdoor pond. The temperatures recorded, in Fahrenheit, are shown below.

Pond Temperature, April 1–5
37°F
35°F
42°F
48°F
46°F

Calculate the standard deviation of these temperatures. Round the answer to the nearest whole number.

4. An app for finding local apartments has a function that analyzes the monthly cost of one-bedroom rentals in a suburb of a major city. This analysis serves to inform its users about the extent of monthly one-bedroom rental cost variability as they search for apartments in this suburb. Refer to the list on the next page for detailed monthly rental cost data.

Calculate the standard deviation of the monthly cost of these rentals to the nearest dollar to gauge the extent of cost variation. Then determine the range to further assess the spread of the monthly rental cost. Round answers to the nearest dollar.

$1,450	$1,750	$1,500	$1,600
$1,675	$1,800	$1,950	$1,700

ANSWERS

CHECK YOUR ANSWERS

1. **A.** Mean: 85.26
   Minimum: 70
   Maximum: 100
   Median: 85
   Mode: 80 and 85
   Range: 30

   B. No outliers

2. **A.** Mean: 1,314
   Minimum: 1,008
   Maximum: 1,923
   Median: 1,227
   Mode: 1,135
   Range: 915

   B. Yes: 1,923

3. 5

4. The standard deviation is approximately 152. The range is 500.

BINOMIAL DISTRIBUTIONS

A **BINOMIAL EXPERIMENT** is an experiment that has exactly two outcomes.

For example, flipping a coin is a binomial experiment because the only two outcomes are heads or tails.

In a binomial experiment, we usually categorize the results as "success" and "failure." In the case of flipping a coin, we can decide which is which.

If we perform a binomial experiment and the probability of success is p, then, because the probabilities of all possible outcomes must add up to 1, the probability of failure is $q = 1 - p$.

For example, when flipping a coin, the probability of heads is $\frac{1}{2}$. Therefore, the probability of tails is $1 - \frac{1}{2} = \frac{1}{2}$.

We will often be interested in performing a binomial experiment multiple times. We refer to each time we perform the experiment as a **TRIAL**.

A **PROBABILITY DISTRIBUTION** is a function that describes the likelihood of obtaining all possible values that a random procedure can take.

A **BINOMIAL DISTRIBUTION** is a probability distribution that deals with experiments having only two possible outcomes: success and failure.

BINOMIAL DISTRIBUTION FORMULA

In a binomial experiment with n trials, where p is the probability of success and $q = 1 - p$ is the probability of failure, the probability of exactly r successes ($0 \le r \le n$) is given by

$$P = {}_nC_r p^r (1 - p)^{n-r}.$$

EXAMPLE: A survey in a recent edition of a high school newsletter confirmed that about 58% of the students at that high school use public transportation to commute to school. Suppose the principal randomly surveys 8 students. What is the probability that exactly 3 of them use public transportation to commute to school? Round the answer to the nearest thousandth.

To answer this question, use the binomial distribution formula.

The "success" in this binomial distribution is a student using public transportation. We are given that the probability of success (p) = 0.58.

In randomly surveying 8 students, the principal is conducting $n = 8$ independent trials.

The probability of getting exactly $r = 3$ successes is:

$P = {}_nC_r p^r(1 - p)^{n-r}$ *binomial distribution formula*

$P = {}_8C_3(0.58)^3(1 - 0.58)^{8-3}$ *Substitute the values into the formula*

$$P = \frac{8!}{3!(8-3)!}(0.58)^3(0.42)^5$$

$P = 56(0.58)^3(0.42)^5$

$P \approx 0.143$ *Round to the nearest thousandth*

So, the probability that *exactly* 3 of the 8 students use public transportation to commute to school, to the nearest thousandth, is 0.143.

EXAMPLE: Draw a histogram of the binomial distribution from the previous example for the high school newspaper survey to show each one of the trials. Then find the probability that at most 3 of the 8 students surveyed use public transportation to commute to school. Round the answer to the nearest thousandth.

To draw the histogram and find the probability, use the binomial distribution formula.

Step 1: Calculate the probability for each value of r from 0 through 8. Round each answer to the nearest ten thousandth.

$P = {}_8C_0(0.58)^0(1 - 0.58)^{8-0}$
≈ 0.001

$P = {}_8C_1(0.58)^1(1 - 0.58)^{8-1}$
≈ 0.0107

$P = {}_8C_2(0.58)^2(1 - 0.58)^{8-2}$
≈ 0.0517

$P = {}_8C_3(0.58)^3(1 - 0.58)^{8-3}$
≈ 0.1428

$P = {}_8C_4(0.58)^4(1 - 0.58)^{8-4}$
≈ 0.2465

$P = {}_8C_5(0.58)^5(1 - 0.58)^{8-5}$
≈ 0.2723

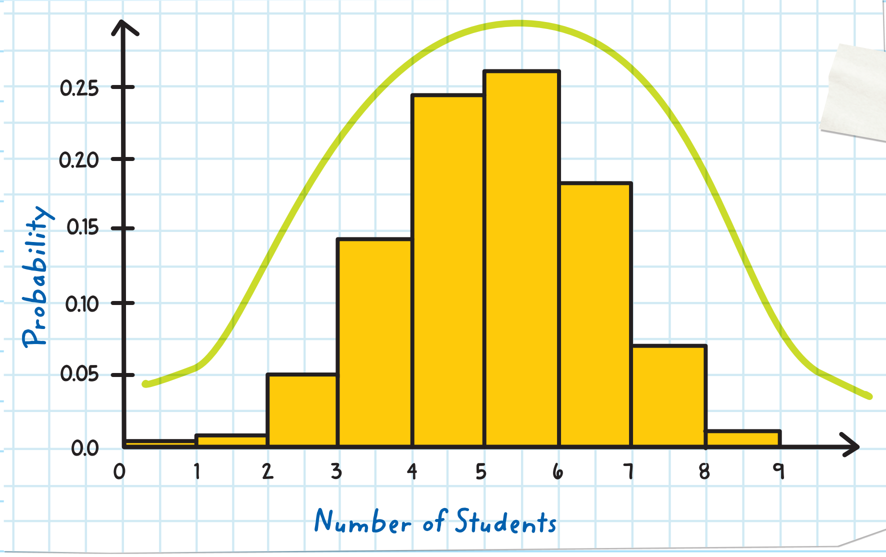
$P = {}_8C_6(0.58)^6(1 - 0.58)^{8-6}$
≈ 0.1880

$P = {}_8C_7(0.58)^7(1 - 0.58)^{8-7}$
≈ 0.0742

$P = {}_8C_8(0.58)^8(1 - 0.58)^{8-8}$
≈ 0.0128

Think: The sum of the probabilities for all the trials for a binomial distribution should always equal 1.

Step 2: Make a histogram.



Notice that the histogram representing the binomial distribution above has a symmetric "bell shape." This tends to happen with binomial distributions. However, when the sample size is small or the probability of success is *not* close to 0.5, the distribution may be skewed left or right.

Step 3: Find the probability that *at most* 3 of the 8 students surveyed use public transportation to commute to school by adding the calculated probabilities for r equal to 3, 2, 1, and 0. Round the answer to the nearest thousandth.

$P(\leq 3) \approx P(3) + P(2) + P(1) + P(0)$

$\approx 0.1428 + 0.0517 + 0.0107 + 0.001 \approx 0.2062$

So, the probability that *at most* 3 of the 8 students surveyed use public transportation to commute to school, to the nearest thousandth, is 0.2062.

Note: Suppose we want the probability that *more than* 3 of the 8 students surveyed use public transportation to commute to school. Since we already know $P(\leq 3)$, we can compute this quite easily:

$P(> 3) = 1 - P(\leq 3) = 1 - 0.2062 = 0.7938.$

Think: Sometimes it may be quicker to find the probability of the complementary event and then subtract that answer from 1.

EXAMPLE: The varsity basketball coach determines that Niles has about a 60% chance of making each free throw. What is the probability that Niles will make more than 3 out of 6 free throws? Round the answer to the nearest thousandth.

To determine the probability that Niles will make more than 3 out of 6 free throws, add the calculated probabilities for *r* equal to 4, 5, and 6.

$P = {}_nC_r p^r(1 - p)^{n-r}$ *binomial distribution formula*

$P(4) = {}_6C_4(0.60)^4(1 - 0.60)^{6-4} \approx 0.3110$

$P(5) = {}_6C_5(0.60)^5(1 - 0.60)^{6-5} \approx 0.1866$

$P(6) = {}_6C_6(0.60)^6(1 - 0.60)^{6-6} \approx 0.0467$

$P(> 3) \approx P(4) + P(5) + P(6)$

$\approx 0.3110 + 0.1866 + 0.0467 \approx 0.5443$

So, the probability that Niles will make *more than* 3 out of 6 free throws is 0.5443.

EXAMPLE: A senior medical sales representative has about a 75% chance of closing a deal with a potential client. What is the probability that the salesperson will close a deal fewer than 3 out of 5 times in one day? Round your answer to the nearest thousandth.

To determine the probability that the medical salesperson will close a deal with a potential client fewer than 3 out of 5 times in one day, add the calculated probabilities for *r* equal to 0, 1, and 2.

$P(0) = {}_5C_0(0.75)^0(1 - 0.75)^{5-0} \approx 0.001$

$P(1) = {}_5C_1(0.75)^1(1 - 0.75)^{5-1} \approx 0.0146$

$P(2) = {}_5C_2(0.75)^2(1 - 0.75)^{5-2} \approx 0.0879$

$P(< 3) \approx P(0) + P(1) + P(2)$

$\approx 0.001 + 0.0146 + 0.0879 \approx 0.1035$

So, the probability that that the medical salesperson will close a deal with a potential client fewer than 3 out of 5 times in one day is 0.1035.

CHECK YOUR KNOWLEDGE

For questions 1 and 2, draw a histogram of the binomial distribution that shows the probability of *exactly r* successes.

1. $p = 0.70$; $n = 5$

2. $p = 0.32$; $n = 7$

3. A minor league baseball player has about an 80% chance of hitting the ball. What is the probability that the player will hit the ball *more than* 4 out of 6 times in a game? Round your answer to the nearest thousandth.

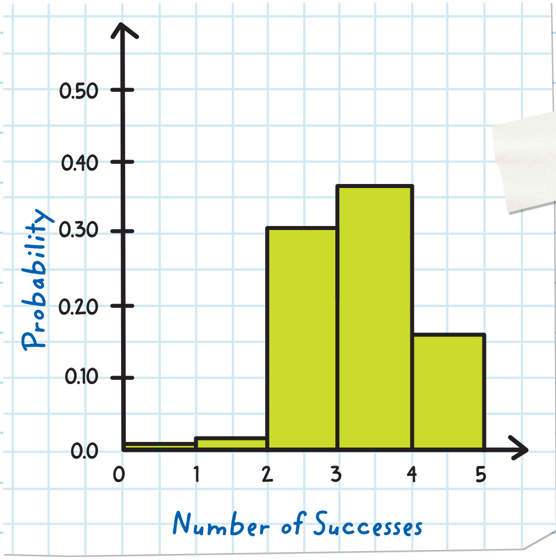
4. A law firm has a 90% success rate of winning a personal injury lawsuit. What is the probability that the law firm will win *fewer than* 5 out of 7 lawsuits? Round your answer to the nearest thousandth.

5. A survey conducted by a major airline confirmed that about 48% of its frequent flyer customers use the airline for business travel. Suppose an airline employee randomly contacts 10 frequent flyer customers. What is the probability that *at most* 6 of these customers use the airline for business travel? Round the answer to the nearest thousandth.

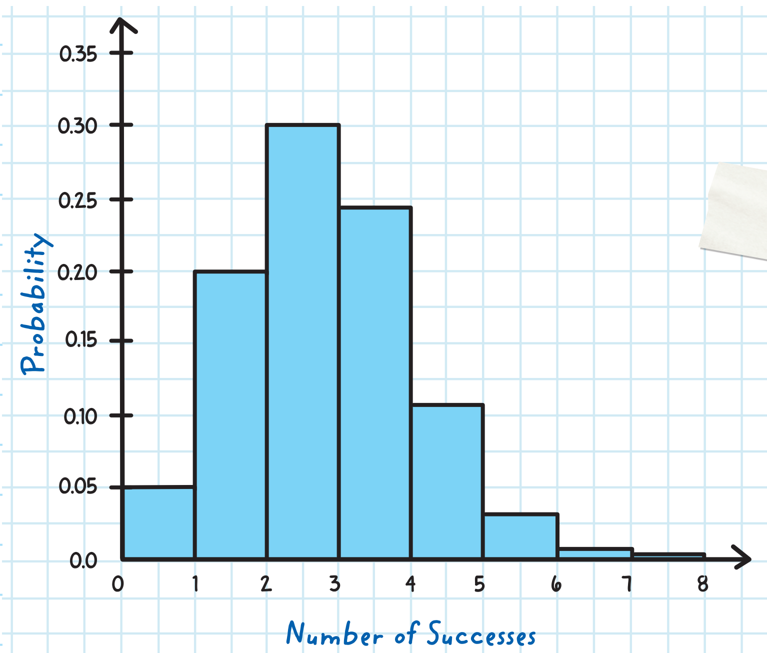
ANSWERS

CHECK YOUR ANSWERS

1.



2.



3. 0.655

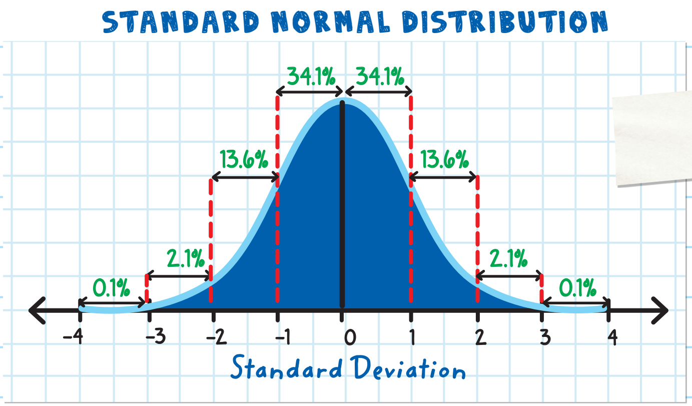
4. 0.026

5. 0.859

NORMAL DISTRIBUTIONS

In a **NORMAL DISTRIBUTION**, data is symmetrically distributed about the mean in a specific way.

The **STANDARD NORMAL CURVE** is bell-shaped:



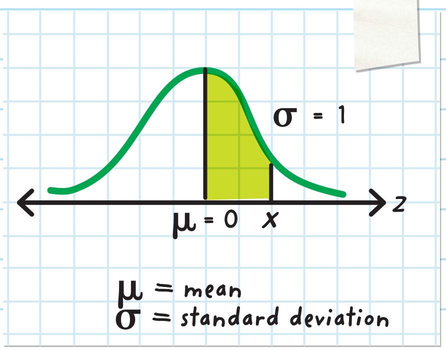
Areas under the standard normal curve can be used to approximate probabilities.

For example, the standard normal curve on the opposite page shows us that the probability that a randomly selected piece of data (which we will represent with the letter z) falls between 0 and 1 is approximately 34.1%. We could write this as $P(0 \leq z \leq 1) \approx 0.341$.

Note: If we use $<$ instead of $\leq$, the probability doesn't change. $P(0 \leq z < 1)$ and $P(0 < z < 1)$ are both also equal to approximately 0.341.

As another example, $P(x \leq -1) \approx 0.5 - 0.341 = 0.159$.

On the next page, there is a table of approximate areas $a(x)$ under the standard normal curve between the mean 0 and the given number of standard deviations x.

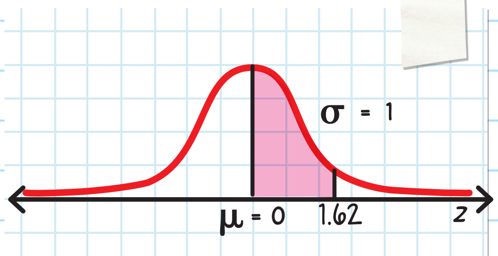


EXAMPLE: Approximate each of the following probabilities by using the standard normal distribution table.

1. $P(0 \le x \le 1.62)$

STANDARD NORMAL DISTRIBUTION TABLE

Z	.00	.01	.02
0.0	.0000	.0040	.0080
0.1	.0398	.0438	.0478
0.2	.0793	.0832	.0871
0.3	.1179	.1217	.1255
0.4	.1554	.1591	.1628
0.5	.1915	.1950	.1985
0.6	.2257	.2291	.2324
0.7	.2580	.2611	.2642
0.8	.2881	.2910	.2939
0.9	.3159	.3186	.3212
1.0	.3413	.3438	.3461
1.1	.3643	.3665	.3686
1.2	.3849	.3869	.3888
1.3	.4032	.4049	.4066
1.4	.4192	.4207	.4222
1.5	.4332	.4345	.4357
1.6	.4452	.4463	.4474
1.7	.4554	.4564	.4573
1.8	.464	.4649	.4656
1.9	.4713	.4719	.4726
2.0	.4772	.4778	.4783



Look for 1.6 on the left side of the distribution table and then .02 on the top of the table.

The entry inside the table at that position is .4474.
So, $P(0 \le x \le 1.62) \approx 0.4474$.

2. $P(x \ge 1.5)$

Z	.00	.01	.02	.03
0.0	.0000	.0040	.0080	.0120
0.1	.0398	.0438	.0478	.0517
0.2	.0793	.0832	.0871	.0910
0.3	.1179	.1217	.1255	.1293
0.4	.1554	.1591	.1628	.1664
0.5	.1915	.1950	.1985	.2019
0.6	.2257	.2291	.2324	.2357
0.7	.2580	.2611	.2642	.2673
0.8	.2881	.2910	.2939	.2967
0.9	.3159	.3186	.3212	.3238
1.0	.3413	.3438	.3461	.3485
1.1	.3643	.3665	.3686	.3708
1.2	.3849	.3869	.3888	.3907
1.3	.4032	.4049	.4066	.4082
1.4	.4192	.4207	.4222	.4236
1.5	.4332	.4345	.4357	.4370
1.6	.4452	.4463	.4474	.4484
1.7	.4554	.4564	.4573	.4582

We can use the table to get $P(0 \le x < 1.5) \approx 0.4332$. (Note that 1.5 = 1.50.)

Since the right half of the graph has an area of 0.5, we can subtract to find the desired area:

$P(x \ge 1.5) \approx 0.5 - 0.4332 = 0.0668$.

3. $P(2.5 \le x \le 3.5)$

We can use the table to get $P(0 \le x < 2.5) \approx 0.4938$ and $P(0 \le x \le 3.5)$.

Z	.00	.01	.02	.03
0.0	.0000	.0040	.0080	.0120
0.1	.0398	.0438	.0478	.0517
0.2	.0793	.0832	.0871	.0910
0.3	.1179	.1217	.1255	.1293
0.4	.1554	.1591	.1628	.1664
0.5	.1915	.1950	.1985	.2019
0.6	.2257	.2291	.2324	.2357
0.7	.2580	.2611	.2642	.2673
0.8	.2881	.2910	.2939	.2967
0.9	.3159	.3186	.3212	.3238
1.0	.3413	.3438	.3461	.3485
1.1	.3643	.3665	.3686	.3708
1.2	.3849	.3869	.3888	.3907
1.3	.4032	.4049	.4066	.4082
1.4	.4192	.4207	.4222	.4236
1.5	.433	.4345	.4357	.4370
1.6	.4452	.4463	.4474	.4484
1.7	.4554	.4564	.4573	.4582
1.8	.4641	.4649	.4656	.4664
1.9	.4713	.4719	.4726	.4732
2.0	.4772	.4778	.4783	.4788
2.1	.4821	.4826	.4830	.4834
2.2	.4861	.4864	.4868	.4871
2.3	.4893	.4896	.4898	.4901
2.4	.4918	.4920	.4922	.4925
2.5	.4938	.4940	.4941	.4943
2.6	.4953	.4955	.4956	.4957
2.7	.4965	.4966	.4967	.4968

Z	.00	.01	.02	.03
0.0	.0000	.0040	.0080	.0120
0.1	.0398	.0438	.0478	.0517
0.2	.0793	.0832	.0871	.0910
0.3	.1179	.1217	.1255	.1293
0.4	.1554	.1591	.1628	.1664
0.5	.1915	.1950	.1985	.2019
0.6	.2257	.2291	.2324	.2357
0.7	.2580	.2611	.2642	.2673
0.8	.2881	.2910	.2939	.2967
0.9	.3159	.3186	.3212	.3238
1.0	.3413	.3438	.3461	.3485
1.1	.3643	.3665	.3686	.3708
1.2	.3849	.3869	.3888	.3907
1.3	.4032	.4049	.4066	.4082
1.4	.4192	.4207	.4222	.4236
1.5	.4332	.4345	.4357	.4370
1.6	.4452	.4463	.4474	.4484
1.7	.4554	.4564	.4573	.4582
1.8	.4641	.4649	.4656	.4664
1.9	.4713	.4719	.4726	.4732
2.0	.4772	.4778	.4783	.4788
2.1	.4821	.4826	.4830	.4834
2.2	.4861	.4864	.4868	.4871
2.3	.4893	.4896	.4898	.4901
2.4	.4918	.4920	.4922	.4925
2.5	.4938	.4940	.4941	.4943
2.6	.4953	.4955	.4956	.4957
2.7	.4965	.4966	.4967	.4968
2.8	.4974	.4975	.4976	.4977
2.9	.4981	.4982	.4982	.4983
3.0	.4987	.4987	.4987	.4988
3.1	.4990	.4991	.4991	.4991
3.2	.4993	.4993	.4994	.4994
3.3	.4995	.4995	.4995	.4996
3.4	.4997	.4997	.4997	.4997
3.5	.4998	.4998	.4999	.4999
3.6	.4998	.4998	.4999	.4999
3.7	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999
3.9	.5000	.5000	.5000	.5000

It follows that $P(2.5 \le x \le 3.5) \approx 0.4998 - 0.4938 = 0.0060$.

4. $P(x \le 2)$

We can use the table to get $P(0 \le x \le 2) \approx 0.4772$.

Z	.00	.01	.02
0.0	.0000	.0040	.0080
0.1	.0398	.0438	.0478
0.2	.0793	.0832	.0871
0.3	.1179	.1217	.1255
0.4	.1554	.1591	.1628
0.5	.1915	.1950	.1985
0.6	.2257	.2291	.2324
0.7	.2580	.2611	.2642
0.8	.2881	.2910	.2939
0.9	.3159	.3186	.3212
1.0	.3413	.3438	.3461
1.1	.3643	.3665	.3686
1.2	.3849	.3869	.3888
1.3	.4032	.4049	.4066
1.4	.4192	.4207	.4222
1.5	.4332	.4345	.4357
1.6	.4452	.4463	.4474
1.7	.4554	.4564	.4573
1.8	.4641	.4649	.4656
1.9	.4713	.4719	.4726
2.0	.4772	.4778	.4783
2.1	.4821	.4826	.4830
2.2	.4861	.4864	.4868
2.3	.4893	.4896	.4898
2.4	.4918	.4920	.4922

Since $P(x < 0) = 0.5$, it follows that $P(x \le 2) \approx 0.5 + 0.4772 = 0.9772$.

Normal distributions have the following properties:

- The curve is symmetric about the mean.
- The mean, median, and mode are all equal.
- The total area under the curve is 1.

Assume that a data set is normally distributed with mean μ and standard deviation σ. Then the **Z-SCORE** for any data value x in the data set is given by the formula

$$z = \frac{x - \mu}{\sigma}.$$

EXAMPLE: Find the probability that a randomly selected x-value is between 19 and 25 in a normal distribution with a mean of 22 and a standard deviation of 3.

List what you know:

mean (μ) = 22
standard deviation (σ) = 3

Find the z-scores corresponding to $x = 19$ and $x = 25$.

Substitute the given values into the z-score formula:

$$z = \frac{x - \mu}{\sigma}$$

$$z_1 = \frac{19 - 22}{3} = \frac{-3}{3} = -1 \qquad z_2 = \frac{25 - 22}{3} = \frac{3}{3} = 1$$

Use the standard normal distribution table to find that $P(0 \leq z \leq 1) \approx 0.3413$.

By symmetry of the normal curve, you also have $P(-1 \leq z \leq 0) \approx 0.3413$.

It follows that $P(-1 \leq z \leq 1) \approx 0.3413 + 0.3413 = 0.6826$.

So, $P(19 \leq x \leq 25) = P\left(\frac{19 - 22}{3} \leq \frac{x - 22}{3} \leq \frac{25 - 22}{3}\right)$
$= P(-1 \leq z \leq 1) \approx 0.6826$.

Therefore, the probability that a randomly selected x-value is between 19 and 25 in a normal distribution with a mean of 22 and a standard deviation of 3 is 0.6826.

EXAMPLE: The wait time for customers to check out at a high-traffic supermarket follows a normal distribution with a mean (μ) of 3 minutes and a standard deviation (σ) of 1.5 minutes. Find the probability that a customer waits more than 4 minutes at this store.

mean (μ) = 3 minutes
standard deviation (σ) = 1.5 minutes

To find $P(x > 4)$, where x = waiting time at the supermarket, first find the z-score corresponding to $x = 4$.

Substitute the given value into the z-score formula:

$$z = \frac{x - \mu}{\sigma}$$

$$z = \frac{4 - 3}{1.5} = \frac{1}{1.5} = \frac{2}{3} \approx 0.67$$

Use the standard normal distribution table to find that $P(0 \leq z \leq 0.67) \approx 0.2486$.

It follows that $P(z > 0.67) \approx 0.5 - 0.2486 = 0.2514$.

So, $P(x > 4) = P\left(\frac{x-3}{1.5} > \frac{4-3}{1.5}\right) \approx P(z > 0.67) \approx 0.2514$.

Therefore, the probability that a customer at the supermarket waits more than 4 minutes is approximately 0.2514.

EXAMPLE: The time it takes for a baked good in a commercial kitchen to reach packaging is normally distributed with a mean time of 20 minutes and a standard deviation of 5 minutes. Find the probability that a baked good takes less than 15 minutes to arrive at packaging.

mean (μ) = 20 minutes
standard deviation (σ) = 5 minutes

To find $P(x < 15)$, where x = time it takes the baked good to reach packaging, first find the z-score corresponding to $x = 15$.

Substitute the given value into the z-score formula:

$$z = \frac{x - \mu}{\sigma}$$

$$z = \frac{15 - 20}{5} = \frac{-5}{5} = -1$$

Use the standard normal distribution table to find that $P(0 \leq z \leq 1) \approx 0.3413$.

By the symmetry of the normal curve, $P(-1 \leq z \leq 0) \approx 0.3413$.

It follows that $P(z < -1) \approx 0.5 - 0.3413 = 0.1587$.

So, $P(x < 15) = P\left(\frac{x-20}{5} < \frac{15-20}{5}\right) \approx P(z < -1) \approx 0.1587$.

Therefore, the probability that the baked good takes less than 15 minutes to reach packaging is approximately 0.1587.

CHECK YOUR KNOWLEDGE

For questions 1 through 3, approximate each of the following probabilities by using the standard normal distribution table.

1. $P(0 \leq z \leq 2.43)$

2. $P(z \geq 3.5)$

3. $P(1.5 \leq z \leq 3)$

4. Find the probability that a randomly selected x-value is greater than 500 in a normal distribution with a mean of 527 and a standard deviation of 112.

5. Find the probability that a randomly selected x-value is between 240 and 270 in a normal distribution with a mean of 527 and a standard deviation of 112.

6. The delivery time for a pharmacy follows a normal distribution with a mean of 30 minutes and a standard deviation of 5 minutes. What is the probability that a delivery can occur in less than 25 minutes?

7. The scores on a standardized exam follow a normal distribution with a mean of 550 and a standard deviation of 150. Find the probability that a randomly selected student scores between 450 and 600 on this exam.

ANSWERS

CHECK YOUR ANSWERS

1. 0.4925

2. 0.00023

3. 0.06545

4. 0.5948

5. 0.0058

6. 0.1587

7. 0.3747

LAWS OF SINES AND COSINES

To find the angle measures and side lengths for triangles *without* a right angle, we use the laws of sines and cosines.

This is the **LAW OF SINES**:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

When asked to solve a triangle, we can use the Law of Sines to find all unknown angle measures and side lengths when given either of the following:

- **two angle measures** and **one side length**
- **two side lengths** and the measure of an **angle opposite** one of those sides

To "solve a triangle" means to find *all* unknown angle measures and *all* unknown side lengths of the given triangle.

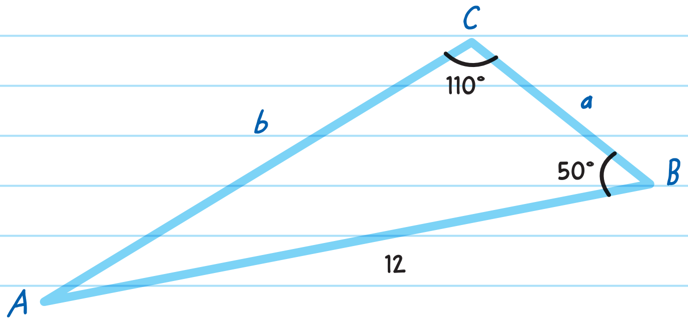
Angle Side Angle (ASA)

Angle Side Angle (ASA) refers to a triangle for which we know the measures of two angles and the length of the side between them.

Angle Angle Side (AAS)

Angle Angle Side (AAS) refers to a triangle for which we know the measures of two angles and the length of a side that is NOT between them.

EXAMPLE: Solve $\triangle ABC$ given that $m\angle B = 50°$, $m\angle C = 110°$, and $c = 12$ feet. Approximate the unknown side lengths to the nearest tenth of a foot.



Note: This triangle satisfies AAS. Therefore, we can use the Law of Sines.

Step 1: Find the measure of the unknown angle using the property that all angle measures of a triangle add to 180°.

$m\angle A + 110° + 50° = 180°$

$m\angle A + 160° = 180°$

$m\angle A = 180° - 160°$

$m\angle A = 20°$

So, $m\angle A = 20°$.

Step 2: Use the Law of Sines to find *a* and *b*.

$$\frac{a}{\sin 20°} = \frac{b}{\sin 50°} = \frac{12}{\sin 110°}$$

Write and solve two equations, each with one variable.

Equation 1:

$$\frac{b}{\sin 50°} = \frac{12}{\sin 110°}$$

Multiply both sides of the equation by sin 50° to isolate the variable.

$$\sin 50° \cdot \frac{b}{\sin 50°} = \frac{12}{\sin 110°} \cdot \sin 50°$$

$$b = \frac{12 \sin 50°}{\sin 110°}$$

Use a calculator to approximate the answer to the nearest tenth.

$b \approx 9.8$ feet

Equation 2:

$$\frac{a}{\sin 20°} = \frac{12}{\sin 110°}$$

Multiply both sides of the equation by sin 20° to isolate the variable.

$$\sin 20° \cdot \frac{a}{\sin 20°} = \frac{12}{\sin 110°} \cdot \sin 20°$$

$$a = \frac{12 \sin 20°}{\sin 110°}$$

Use a calculator to approximate the answer to the nearest tenth.

$a \approx 4.4$ feet

So, the unknown angle measure is $m\angle A = 20°$, and the unknown side lengths are $a \approx 4.4$ feet and $b \approx 9.8$ feet.

Side Side Angle (SSA)

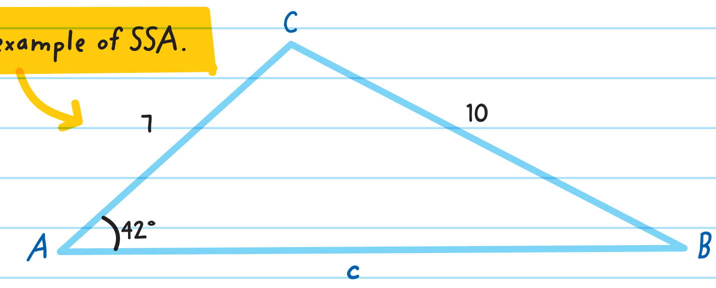
Side Side Angle (SSA) refers to a triangle for which we know the lengths of two sides and the measure of an angle NOT between them.

Solving an SSA triangle has three possibilities: There can be 0, 1, or 2 solutions. For this reason, SSA is commonly called the *ambiguous* case.

Each of the following examples illustrates how to determine if a given SSA triangle has 0, 1, or 2 solutions.

EXAMPLE: Solve $\triangle ABC$ given that $m\angle A = 42°$, $a = 10$ feet, and $b = 7$ feet. Approximate the unknown angle measures to the nearest degree and the unknown side length to the nearest tenth of a foot.

This is an example of SSA.



Step 1: Use the Law of Sines to find the measure of ∠B.

The Law of Sines gives the following equation:

$$\frac{\sin 42°}{10} = \frac{\sin B}{7}$$

NOTE: Here we wrote the Law of Sines using reciprocals to make the computations simpler.

Multiply each side of the equation by 7 to isolate sin B.

$$7 \cdot \frac{\sin 42°}{10} = \frac{\sin B}{7} \cdot 7$$

$$\sin B = \frac{7 \sin 42°}{10}$$

$$m\angle B = \sin^{-1}\left(\frac{7 \sin 42°}{10}\right)$$

Use a calculator to approximate the answer to the nearest degree.

$m\angle B \approx 28°$

Since this is the AMBIGUOUS CASE, there could be another solution. We get this possible solution by subtracting the previous solution from 180°.

$$180° - 28° = 152°$$

We need to check if this possible solution is an actual solution. Since 42 + 152 = 194 > 180, such a triangle cannot exist. Therefore, reject this possibility.

So, △ABC has just *one solution*.

Step 2: Find the approximate measure of the third angle C by using the property that all angle measures of a triangle sum to 180°.

$42° + 28° + m\angle C = 180°$

$m\angle C = 180° - 70°$

$m\angle C = 110°$

So, $m\angle C \approx 110°$.

Step 3: Use the Law of Sines to find c.

$$\frac{10}{\sin 42°} = \frac{c}{\sin 110°}$$

Multiply both sides of the equation by sin 110° to isolate the variable.

$$\sin 110° \cdot \frac{10}{\sin 42°} = \frac{c}{\sin 110°} \cdot \sin 110°$$

$$c = \frac{10 \sin 110°}{\sin 42°}$$

Use a calculator to approximate the answer to the nearest tenth.

$c \approx 14$

So, the unknown angle measures are $m\angle B \approx 28°$ and $m\angle C \approx 110°$, and the unknown side length is $c \approx 14$ feet.

EXAMPLE: Solve $\triangle ABC$ given that $m\angle A = 30°$, $a = 5$ cm, and $b = 8$ cm. Approximate the unknown angle measures to the nearest degree and the unknown side length to the nearest tenth of a centimeter.

The triangle satisfies SSA.

Step 1: Use the Law of Sines to find the measure of $\angle B$.

The Law of Sines gives the following equation:

$$\frac{\sin 30°}{5} = \frac{\sin B}{8}$$

Multiply both sides of the equation by 8 to isolate sin B.

$$\sin B = \frac{8 \sin 30°}{5} = \frac{8}{5} \cdot \frac{1}{2} = \frac{4}{5}$$

Note that there are two angles between 0 and π (or between 0 and 180°) that satisfy this equation. Indeed, if $-1 < y < 1$, then the equation $\sin B = y$ always has two solutions between 0 and π.

To find the first angle, evaluate the following:

$$m\angle B = \sin^{-1} \frac{4}{5}$$

Use a calculator to approximate the answer to the nearest degree:

$$m\angle B \approx 53°$$

To find the second angle, subtract the measure of the first angle from π (or 180°).

$$m\angle B = 180° - 53°$$
$$m\angle B \approx 127°$$

So, $m\angle B \approx 53°$ or $127°$.

If 53° is added to the given measure of angle A, 30°, the sum is less than 180°. The same is true for 127°. Therefore, both measures lead to solutions, and so there are two.

Step 2: Find the approximate measure of the third angle C by using the property that all angle measures of a triangle sum to 180°.

Because there are two solutions for $\triangle ABC$, one with $m\angle B \approx 53°$ and another with $m\angle B \approx 127°$, we need to find two measures for angle C.

$m\angle B \approx 53°$

$30° + 53° + m\angle C = 180°$

$m\angle C = 180° - 83°$

$m\angle C = 97°$

$m\angle C \approx 97°$

$m\angle B \approx 127°$

$30° + 127° + m\angle C = 180°$

$m\angle C = 180° - 157°$

$m\angle C = 23°$

$m\angle C \approx 23°$

Step 3: Use the Law of Sines to find c.

Again, because there are two solutions for $\triangle ABC$, we need to solve two equations, one for $m\angle C \approx 97°$ and one for $m\angle C \approx 23°$.

$m\angle C \approx 97°$

$$\frac{5}{\sin 30°} = \frac{c}{\sin 97°}$$

Multiply both sides of the equation by sin 97° to isolate the variable.

$$\sin 97° \cdot \frac{5}{\sin 30°} = \frac{c}{\sin 97°} \cdot \sin 97°$$

$$c = \frac{5 \sin 97°}{\sin 30°}$$

Use a calculator to approximate the length to the nearest tenth.

$c \approx 9.9$

$m\angle C \approx 23°$

$$\frac{5}{\sin 30°} = \frac{c}{\sin 23°}$$

Multiply both sides of the equation by sin 23° to isolate the variable.

$$\sin 23° \cdot \frac{5}{\sin 30°} = \frac{c}{\sin 23°} \cdot \sin 23°$$

$$c = \frac{5 \sin 23°}{\sin 30°}$$

Use a calculator to approximate the answer to the nearest tenth.

$c \approx 3.9$

So, there are two solutions.

First solution:	Second solution:
$m\angle B \approx 53°$	$m\angle B \approx 127°$
$m\angle C \approx 97°$	$m\angle C \approx 23°$
$c \approx 9.9$ cm	$c \approx 3.9$ cm

EXAMPLE: Solve $\triangle ABC$ given that $m\angle A = 58°$, $a = 2.5$ m, and $b = 4$ m. Approximate the unknown angle measures to the nearest degree and the unknown side length to the nearest tenth of a meter.

The triangle satisfies SSA.

Step 1: Use the Law of Sines to find the measure of $\angle B$.

The Law of Sines gives the following equation:

$$\frac{\sin 58°}{2.5} = \frac{\sin B}{4}$$

Multiply both sides of the equation by 4 to isolate sin B.

$$\sin B = \frac{4 \sin 58°}{2.5}$$

Use a calculator to approximate the answer.

$\sin B \approx 1.4.$

Since there is no angle whose sine is 1.4, this equation has *no solution*.

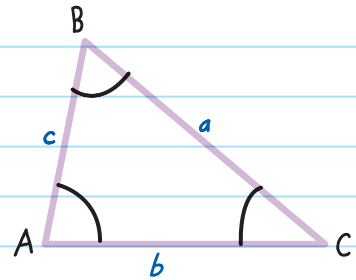
So, no triangle exists with the given information.

LAW OF COSINES

This is the **LAW OF COSINES**:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

When asked to solve a triangle, we can use the Law of Cosines to find all unknown angle measures and side lengths when we are given either of the following:



- two side lengths and the measure of the included angle

  or

- all three side lengths

To solve a triangle completely, we may need to use the Law of Cosines together with the Law of Sines.

Side Angle Side (SAS)

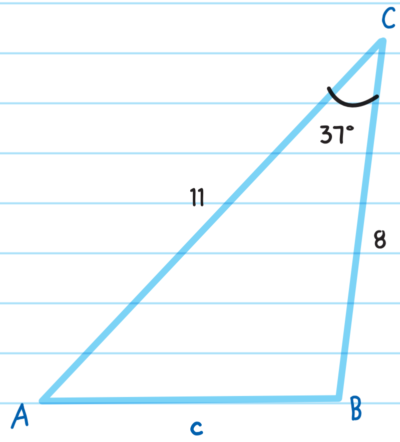
Side Angle Side (SAS) refers to a triangle for which we know the lengths of two sides and the measure of the angle between them.

Side Side Side (SSS)

Side Side Side (SSS) refers to a triangle for which we know the lengths of all three sides of a triangle.

In these cases, there is always EXACTLY one solution.

EXAMPLE: Solve $\triangle ABC$ given that $m\angle C = 37°$, $a = 8$ inches, and $b = 11$ inches. Approximate the unknown angles to the nearest degree and the unknown side length to the nearest tenth of an inch.



This triangle satisfies SAS.

Step 1: Use the Law of Cosines to find c.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 8^2 + 11^2 - 2 \cdot 8 \cdot 11 \cos 37°$$

$$c^2 = 185 - 176 \cos 37°$$

$$c^2 \approx 44.44015023$$

$$c \approx 6.7$$

Step 2: Use the Law of Sines to approximate $m\angle A$.

Note: It would NOT be a good idea to use the Law of Sines to find $m\angle B$. The reason is that $\angle B$ is the largest angle (because it's opposite the longest side), so it could be acute (a Quadrant I angle) or obtuse (a Quadrant II angle), whereas $\angle A$ *must* be acute.

$$\frac{\sin 37°}{6.7} = \frac{\sin A}{8}$$

Multiply both sides of the equation by 8 to isolate sin A.

$$\sin A = \frac{8 \sin 37°}{6.7}$$

$$m\angle A = \sin^{-1}\left(\frac{8 \sin 37°}{6.7}\right)$$

Use a calculator to approximate the answer to the nearest degree.

$$m\angle A \approx 46°$$

Step 3: Find the measure of the third angle, $\angle B$, by using the property that all angle measures of a triangle sum to 180°.

$46° + 37° + m\angle B = 180°$

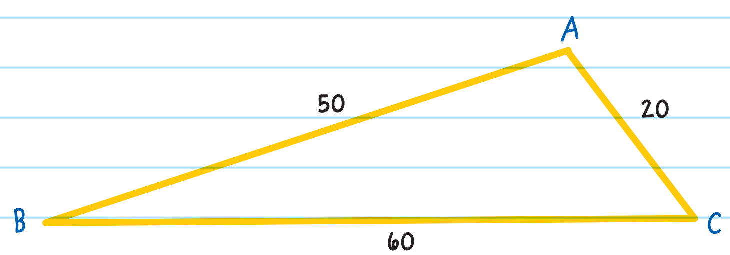
$m\angle B = 180° - 83°$

$m\angle B = 97°$

$m\angle B \approx 97°$

So, the unknown angle measures are $m\angle A \approx 46°$ and $m\angle B \approx 97°$, and the unknown side length is $c \approx 6.7$ inches.

EXAMPLE: Solve $\triangle ABC$ given that $a = 60$ cm, $b = 20$ cm, and $c = 50$ cm. Approximate the angles to the nearest degree.



This triangle satisfies SSS.

Step 1: Use the Law of Cosines to find one of the angle measures. We will find $m\angle B$.

$b^2 = a^2 + c^2 - 2ac\cos B$

$-2ac\cos B = b^2 - a^2 - c^2$

$\cos B = \dfrac{b^2 - a^2 - c^2}{-2ac}$

$= \dfrac{20^2 - 60^2 - 50^2}{-2 \cdot 60 \cdot 50}$

$= \dfrac{-5{,}700}{-6{,}000}$

$= \dfrac{19}{20}$

$m\angle B = \cos^{-1}\dfrac{19}{20}$

Use a calculator to approximate the answer to the nearest degree.

$m\angle B \approx 18°$

Step 2: Use the Law of Sines or the Law of Cosines to find another angle measure.

We will find $m\angle A$ using the Law of Cosines.

Note: If the Law of Sines is used instead, make sure to find m∠C because ∠C must be an acute angle.

$a^2 = b^2 + c^2 - 2bc\cos A$

$-2bc\cos A = a^2 - b^2 - c^2$

$\cos A = \dfrac{a^2 - b^2 - c^2}{-2bc}$

$= \dfrac{60^2 - 20^2 - 50^2}{-2 \cdot 20 \cdot 50}$

$= \dfrac{700}{-2{,}000}$

$= -\dfrac{7}{20}$

$m\angle A = \cos^{-1}\left(-\dfrac{7}{20}\right)$

Use a calculator to approximate the answer to the nearest degree.

$m\angle A \approx 110°$

Step 3: Find the measure of the third angle, ∠C, by using the property that all angle measures of a triangle sum to 180°.

$110° + 18° + m\angle C \approx 180°$

$m\angle C \approx 180° - 128°$

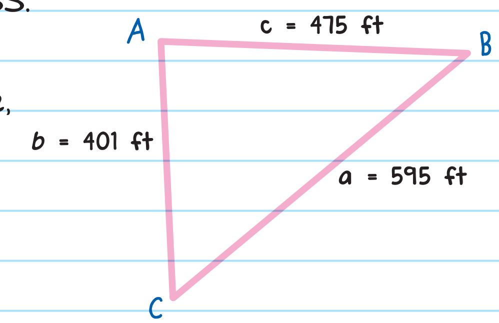
$m\angle C \approx 52°$

So, the angle measures are m∠A ≈ 110°, m∠B ≈ 18°, and m∠C ≈ 52°.

EXAMPLE: An engineer for the Carington City highways is inspecting a triangular median and needs to compute its angle measures. The lengths of the sides of the triangular median are 595 feet, 401 feet, and 475 feet. Approximate the angle measures to the nearest tenth of a degree.

This triangle satisfies SSS.

Step 1: Draw the triangle, giving a letter name to each of the side lengths and angles.



Step 2: Use the Law of Cosines to find one of the angle measures. We will find m∠A.

$a^2 = b^2 + c^2 - 2bc\cos A$

$595^2 = 401^2 + 475^2 - 2 \cdot 401 \cdot 475\cos A$

$-2 \cdot 401 \cdot 475\cos A = 595^2 - 401^2 - 475^2$

$$\cos A = \frac{595^2 - 401^2 - 475^2}{-2 \cdot 401 \cdot 475}$$

$$\cos A = \frac{-32{,}401}{-380{,}950}$$

$$m\angle A = \cos^{-1}\frac{32{,}401}{380{,}950}$$

Use a calculator to approximate the answer to the nearest tenth of a degree.

$$m\angle A \approx 85.1°$$

Step 3: Use the Law of Sines or Cosines to find another angle measure. We will find $m\angle B$ using the Law of Sines.

$$\frac{\sin 85.1°}{595} = \frac{\sin B}{401}$$

Multiply both sides of the equation by 401 to isolate sin B.

$$\sin B = \frac{401 \sin 85.1°}{595}$$

$$m\angle B = \sin^{-1}\left(\frac{401 \sin 85.1°}{595}\right)$$

Use a calculator to approximate the answer to the nearest tenth of a degree.

$$m\angle B \approx 42.2°$$

Step 4: Find the measure of the third angle, $\angle C$, by using the property that all angle measures of a triangle sum to 180°.

$$85.1° + 42.2° + m\angle C = 180°$$

$$m\angle C = 180° - 127.3°$$

$$m\angle C = 52.7°$$

$$m\angle C \approx 52.7°$$

So, the angle measures are $m\angle A \approx 85.1°$, $m\angle B \approx 42.2°$, and $m\angle C \approx 52.7°$.

HERON'S FORMULA

The Laws of Sines and Cosines have many theoretical applications. As one example, the Law of Cosines can be used to derive **HERON'S FORMULA**. This formula allows us to compute the area of any triangle knowing only the three side lengths of the triangle.

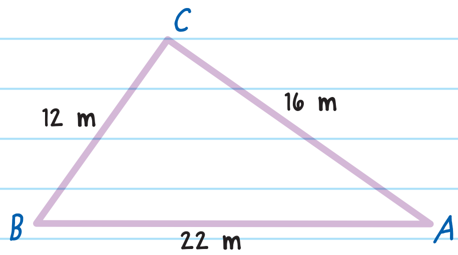
$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a + b + c)$.

s here is known as the **SEMIPERIMETER** (half of the perimeter) of the triangle.

EXAMPLE: Find the area of $\triangle ABC$. Round the area to the nearest tenth of a square meter.

Since three side lengths are given, find the area using Heron's Formula.



Step 1: Find the semiperimeter.

Let $a = 12$, $b = 16$, and $c = 22$.

$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(12 + 16 + 22) = \frac{1}{2} \cdot 50 = 25$$

Step 2: Apply Heron's Formula.

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{25(25-12)(25-16)(25-22)}$$

$$= \sqrt{25 \cdot 13 \cdot 9 \cdot 3}$$

$$= \sqrt{8{,}775}$$

$$\approx 93.7 \text{ m}^2$$

So, the area of $\triangle ABC$ is approximately 93.7 m^2.

CHECK YOUR KNOWLEDGE

For questions 1 through 8, solve $\triangle ABC$ (find *all* unknown angle measures and *all* unknown side lengths). Approximate the unknown angle measures to the nearest degree and the unknown side lengths to the nearest integer.

Remember: There can be 0, 1, or 2 triangles with the given side lengths and angle measures.

1. $m\angle A = 110°$, $m\angle C = 30°$, $b = 15$ cm

2. $a = 14$ inches, $m\angle C = 65°$, $b = 12$ inches

3. $a = 67$ m, $c = 93$ m, $m\angle A = 34°$

4. $a = 7$ feet, $m\angle A = 37°$, $m\angle B = 76°$

5. $b = 4$ mm, $m\angle C = 58°$, $c = 2.5$ mm

6. $a = 16$ feet, $b = 32$ feet, $c = 39$ feet

7. $b = 4$ mm, $m\angle A = 45°$, $c = 2$ mm

8. A conservation group acquired a triangular piece of land to help preserve plants and wildlife. The side lengths of the land are 153 miles, 175 miles, and 201 miles. To visualize the area, the group hired an artist to render the land from an overhead view. To complete the drawing, the artist must determine the measure of the three angles created where the three boundary lines of the land meet. What are the approximate angle measures to the nearest degree?

9. The courtyard in a new athletic complex is triangular. One side of the courtyard is 25 feet long and another is 19 feet long. The angle opposite the 25-foot side is 95°. To the nearest tenth of a foot, how many feet long is the third side of the courtyard?

For questions 11 and 12, find the area of $\triangle ABC$. Approximate the area to the nearest inch or centimeter.

10. $a = 47$ inches, $b = 30$ inches, $c = 62$ inches

11. $a = 21$ cm, $m\angle B = 70°$, $c = 41$ cm

ANSWERS

CHECK YOUR ANSWERS

1. $m\angle B = 40°$, $a \approx 22$ cm, $c \approx 12$ cm

2. $m\angle A \approx 64°$, $m\angle B \approx 51°$, $c \approx 14$ in

3. Two solutions:
   $m\angle B \approx 95°$, $m\angle C \approx 51°$, $b \approx 119$ m
   $m\angle B \approx 17°$, $m\angle C \approx 129°$, $b \approx 35$ m

4. $b \approx 11$ feet, $c \approx 11$ feet, $m\angle C = 67°$

5. No solution

6. $m\angle A \approx 23°$, $m\angle B \approx 53°$, $m\angle C \approx 104°$

7. $a \approx 3$ mm, $m\angle B \approx 106°$, $m\angle C \approx 29°$

8. 57°, 75°, 47°

9. 14.7 feet

10. 681 inches^2

11. 405 cm^2